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List Edge and List Total Colourings of Multigraphs

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This paper exploits the remarkable new method of Galvin (*J. Combin. Theory Ser. B* **63** (1995), 153–158), who proved that the list edge chromatic number $\chi'_{\text{list}}(G)$ of a bipartite multigraph G equals its edge chromatic number $\chi'(G)$. It is now proved here that if every edge $e = uw$ of a bipartite multigraph G is assigned a list of at least $\max\{d(u), d(w)\}$ colours, then G can be edge-coloured with each edge receiving a colour from its list. If every edge $e = uw$ in an arbitrary multigraph G is assigned a list of at least $\max\{d(u), d(w)\} + \lfloor \frac{1}{2} \min\{d(u), d(w)\} \rfloor$ colours, then the same holds; in particular, if G has maximum degree $\Delta = \Delta(G)$ then $\chi'_{\text{list}}(G) \leqslant \lfloor \frac{3}{2}\Delta \rfloor$. Sufficient conditions are given in terms of the maximum degree and maximum average degree of G in order that $\chi'_{\text{list}}(G) = \Delta$ and $\chi''_{\text{list}}(G) = \Delta + 1$. Consequences are deduced for planar graphs in terms of their maximum degree and girth, and it is also proved that if G is a simple planar graph and $\Delta \geqslant 12$ then $\chi'_{\text{list}}(G) = \Delta$ and $\chi''_{\text{list}}(G) = \Delta + 1$. © 1997 Academic Press

1. INTRODUCTION

Let $G = (V, E)$ be a multigraph with vertex-set $V(G) = V$ and edge-set $E(G) = E$. If $v \in V$ and $X \subseteq V$, we write $N(X)$ for the set of neighbours of vertices in X, and $N(v) := N(\{v\})$. We write $d(v) = d_G(v)$ for the degree of

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v in G, and $\Delta(G)$ and $\delta(G)$ for the maximum and minimum degrees in G; clearly $d(v) \geqslant |N(v)|$, with inequality possible.

Let $f\colon V \cup E \to \mathbb{N}$ be a function into the positive integers. We say that G is *totally-f-choosable* if, whenever we are given sets ("lists") A_x of "colours" with $|A_x| = f(x)$ for each $x \in V \cup E$, we can choose a colour $c(x) \in A_x$ for each element x so that no two adjacent vertices or adjacent edges have the same colour, and no vertex has the same colour as an edge incident with it. The *list total chromatic number* $\chi''_{\text{list}}(G)$ of G is the smallest integer k such that G is totally-f-choosable when $f(x) = k$ for each x. The *list* (*vertex*) *chromatic number* $\chi_{\text{list}}(G)$ of G and the *list edge chromatic number* (or *list chromatic index*) $\chi'_{\text{list}}(G)$ of G are defined similarly in terms of colouring vertices alone, or edges alone, respectively; and so are the concepts of *vertex-f-choosability* and *edge-f-choosability*. The ordinary vertex, edge, and total chromatic numbers of G are denoted by $\chi(G)$, $\chi'(G)$, and $\chi''(G)$.

It is easy to see (by considering complete-bipartite graphs, cf. [10, 30]) that there is no bound for $\chi_{\text{list}}(G)$ in terms of $\chi(G)$ in general. In contrast, part (a) of the following conjecture was formulated independently by Vizing, by Gupta, by Albertson and Collins, and by Bollobás and Harris (see [13] or [16]), and it is well known as the *list colouring conjecture*; we have not seen part (b) before, but it seems to us a natural extension of part (a).

Conjecture A. If G is a multigraph then

$$\text{(a)}\quad \chi'_{\text{list}}(G) = \chi'(G), \qquad \text{(b)}\quad \chi''_{\text{list}}(G) = \chi''(G).$$

The figure $\lfloor \frac{3}{2}\Delta \rfloor$ in the following theorem is best possible in view of the "Shannon triangle," obtained by replacing the three edges of K_3 by sheaves of $\lfloor \frac{1}{2}\Delta \rfloor$, $\lfloor \frac{1}{2}\Delta \rfloor$, and $\lceil \frac{1}{2}\Delta \rceil$ parallel edges, respectively.

THEOREM A. *If G is a multigraph with maximum degree Δ, then*

(a) (Shannon, [25]) $\chi'(G) \leqslant \lfloor \frac{3}{2}\Delta \rfloor$,

(b) (Kostochka [20–23]) $\chi''(G) \leqslant \lfloor \frac{3}{2}\Delta \rfloor$ if $\Delta \geqslant 4$.

For (simple) graphs we have the following well-known theorem and conjecture; in each case, the lower bound is obvious. (In the rest of this section we shall write Δ as a shorthand for $\Delta(G)$.)

THEOREM B (Vizing [29], Gupta [12]). *If G is simple then $\chi'(G) = \Delta$ or $\Delta + 1$.*

Conjecture B (Vizing [29], Behzad [1]). If G is simple then $\chi''(G) = \Delta + 1$ or $\Delta + 2$.

In view of Theorems A and B, Conjecture A(a) would imply that $\chi'_{\text{list}}(G) \leqslant \lfloor \frac{3}{2}\Delta \rfloor$ for multigraphs, with $\chi'_{\text{list}}(G) \leqslant \Delta+1$ if G is simple. In Section 3 we shall prove the first of these results. The best upper bound previously known for multigraphs seems to have been $\chi'_{\text{list}}(G) \leqslant \frac{9}{5}\Delta$ by Hind [15], although we should also mention the asymptotic result of Kahn [19] that, for each $\varepsilon > 0$, there exists $K(\varepsilon)$ such that for each multigraph G with $\Delta(G) > K(\varepsilon)$ we have $\chi'_{\text{list}}(G) < (1+\varepsilon)\,\chi'(G)$. (More significantly, Kahn proves this with $\chi'(G)$ replaced by the fractional chromatic index $\chi'^*(G)$.) For simple graphs the best upper bounds known seem to be $\chi'_{\text{list}}(G) \leqslant \frac{7}{4}\Delta + \lceil 25 \log \Delta \rceil$ by Bollobás and Hind [2] and $\chi'_{\text{list}}(G) \leqslant \Delta + c\Delta^{2/3}\sqrt{\log \Delta}$ by Häggkvist and Janssen [14], for some constant $c > 0$; this last result bounds the error term that was left unspecified by Kahn [17, 18], who was the first to prove $\chi'_{\text{list}}(G) \leqslant \Delta + o(\Delta)$.

Galvin [11] introduced a remarkable new technique and proved Conjecture A(a) for bipartite multigraphs. In Section 2 we reproduce Galvin's argument in a more "elementary" formulation (see also [26]), and we obtain a nonuniform analogue of it in which the lists need not have the same cardinality. Specifically, the main result of this paper (Theorem 3) is that if $f(e) = \max\{d(u), d(w)\}$ for each edge $e = uw$ in a bipartite multigraph G, then G is edge-f-choosable.

In Section 3 we apply this result so as to obtain an analogous nonuniform result for nonbipartite multigraphs (Theorem 4), which implies the result $\chi'_{\text{list}}(G) \leqslant \lfloor \frac{3}{2}\Delta \rfloor$ mentioned above; and we draw the obvious conclusions for list total colourings (Corollaries 5.1 and 5.2). In Section 4 we use Theorem 4 to show that $\chi''(G) \leqslant \lfloor \frac{3}{2}\Delta \rfloor + 1$ (Theorem 6); this is weaker than Theorem A(b), but the proof is much shorter. In Section 5 we use Theorem 3 to obtain conditions involving Δ and the maximum average degree of a multigraph G in order that $\chi'_{\text{list}}(G) = \Delta$ and $\chi''_{\text{list}} = \Delta + 1$ (Theorem 7), and we deduce that these same conclusions hold for graphs with specified maximum degree and girth in the plane and some other surfaces (Corollaries 7.1 and 7.2). Finally, in Section 6 we prove that the same conclusions hold if G is simple and planar and $\Delta \geqslant 12$ (Theorem 9). The paper ends with two conjectures (Section 7).

2. LIST EDGE COLOURINGS OF BIPARTITE MULTIGRAPHS

Throughout this section, $G = (V, E)$ will be a bipartite multigraph with partite sets U, W, so that $V = U \cup W$. Let f, g: $E \to \mathbb{N} \cup \{0\}$ be two functions into the nonnegative integers; we call $f(e)$ the *supply* and $g(e)$ the *demand* of edge e. Modifying [11], we say that G is *edge*-$(f:g)$-*choosable* if, whenever we are given sets ("lists") A_e of "colours" with $|A_e| = f(e)$ for each $e \in E$, we can choose subsets $B_e \subseteq A_e$ with $|B_e| = g(e)$ so that $B_e \cap B_{e'} = \varnothing$ whenever e, e' are adjacent. For $t \in \mathbb{N}$, we say that G is *edge-*$(f : t)$*-choosable* if it is edge-$(f : g)$-choosable where $g(e) := t$ for each e, so that edge-$(f : 1)$-choosable means the same as edge-f-choosable.

Let $c: E \to \mathbb{Z}$ be a (proper) edge-colouring of G, to be chosen carefully later. We say that an edge e *sees* an edge $e' \neq e$ if e, e' are adjacent at a vertex v and $c(e) > c(e')$ if $v \in U$ and $c(e) < c(e')$ if $v \in W$. (If e, e' are parallel edges then e sees e' and e' sees e.) If S, $T \subseteq E$, we say that S *sees* T if every edge of $S \backslash T$ sees at least one edge of T. Let $V(S)$ denote the set of end-vertices of edges in S. The following lemma is effectively the last paragraph of the proof of Theorem 4.1 in [11].

LEMMA 1.1 [11]. *If $S \subseteq E$, then there is a matching $M \subseteq S$ such that S sees M.*

Proof [11]. We use induction on $|S|$, noting that the result is trivial if $|S| \leqslant 1$. For each vertex $v \in V(S)$, let e_v be the edge of S at v for which $c(e_v)$ is smallest, and let $M := \{e_u : u \in V(S) \cap U\}$. If M is a matching we are finished. If not, let e_u, $e_{u'}$ meet at w in W. Without loss of generality $e_u \neq e_w$, so that $c(e_u) > c(e_w)$. By the induction hypothesis, $S \backslash \{e_w\}$ contains a matching M' that it sees. Since e_u sees no other edge of S at u, either $e_u \in M'$ or e_u sees an edge $e' \in M'$ at w. But e_w sees e_u and hence e' at w, and so in either case S sees M'. ∎

In view of Lemma 1.1, the following theorem is a special case of Lemma 2.1 in [11]; we include its proof for completeness.

THEOREM 1 [11]. *If, for each edge e, $f(e) - g(e)$ is at least as large as the sum of the demands of all edges that e sees, then G is edge-$(f : g)$-choosable.*

Proof [11]. Suppose we are given lists A_e with $|A_e| = f(e)$ for each $e \in E$. Given a colour c, let $S_c := \{e \in E : c \in A_e \text{ and } g(e) > 0\}$. We prove the result by induction on $\Sigma := \sum_{e \in E} g(e)$. If $\Sigma = 0$ then the result clearly holds. So we may suppose $\Sigma > 0$ and choose c so that $S_c \neq \varnothing$. By Lemma 1.1, S_c contains a matching M that it sees. For each $e \in M$, give colour c to e and reduce $g(e)$ by one. For each $e \in S_c$, remove c from S_e and reduce $f(e)$ by one. The hypothesis of the theorem remains true, and so the result follows by induction. ∎

Let $m_c(e)$ denote the number of edges that e sees according to the colouring c.

COROLLARY 1.1. *If, for each edge e of G, $m_c(e) < f(e)$, then G is edge-$(tf : t)$-choosable for each $t \in \mathbb{N}$; in particular, G is edge-f-choosable.*

Corollary 1.1 clearly follows immediately from Theorem 1, and it in turn immediately implies Galvin's theorem that $\chi'_{\text{list}}(G)=\chi'(G)$, since if $c: E\to\mathbb{Z}$ is any (proper) edge-colouring of G with $\chi'(G)$ colours then evidently $m_c(e)<\chi'(G)$ for each edge e. However, we can get stronger consequences from Corollary 1.1 if we choose c more carefully. Theorem 2 is a simple result of this kind; Theorem 3 is stronger (and implies Theorem 2(b)), but its proof is longer. The notation $e=uw$ implies $u\in U$, $w\in W$.

THEOREM 2. *Let $f(e):=d(w)$ for each edge $e=uw$. Then*

(a) *G is edge-f-choosable if and only if G has a (proper) edge-colouring $c: E\to\mathbb{N}$ in which the edges at each vertex w in W are coloured with the colours $1, ..., d(w)$.*

(b) *If $d(u)\leqslant d(w)$ for every edge $e=uw$, then G is edge-f-choosable.*

Proof. "Only if" holds in (a) because we could choose the list $A_c:=\{1, ..., d(w)\}$ for each edge $e=vw$. "If" holds by Corollary 1.1, since it is clear that with the hypothesized edge-colouring c, $m_c(e)<d(w)$ for each edge $e=uw$.

The truth of (b) will now follow from the following (iterative) construction of a colouring c as described in (a). Let $\Delta:=\Delta(G)$. By the König–Hall theorem, it is easy to see that there is a matching M in G such that $V(M)$ includes all vertices of degree Δ. Moreover, vertices of degree Δ in U are adjacent only to other vertices of degree Δ, and so we can choose M so that $V(M)\cap W$ comprises precisely the vertices of degree Δ in W. Define $c(e):=\Delta$ for each $e\in M$, remove the edges of M from G, and iterate. ∎

For Theorem 3 we shall need the following simple lemma.

LEMMA 3.1. *If $|U|\leqslant|W|$ and W contains no isolated vertices, then G contains a nonempty matching M such that whenever $e=uw\in E$ and $w\in V(M)$, then $u\in V(M)$.*

Proof. Clearly $|N(W)|\leqslant|U|\leqslant|W|$. Let X be a minimal nonempty subset of W such that $|N(X)|\leqslant|X|$. Then $|N(X)|=|X|$ and there is a matching M such that $V(M)=X\cap N(X)$. (If $|X|=1$, this holds because W contains no isolated vertices. If $|X|\geqslant 2$, it holds by the König–Hall theorem and since $|N(Y)|>|Y|$ whenever $\varnothing\subsetneqq Y\subsetneqq X$.) Clearly M has the required property. ∎

Theorem 3 is the main result of this paper. It is the natural two-sided generalization of Theorem 2(b).

THEOREM 3. *G is edge-f-choosable, where $f(e):=\max\{d(u), d(w)\}$ for each edge $e=uw$. In fact, G is edge-$(tf:t)$-choosable, for each $t\in\mathbb{N}$.*

Proof. We prove by induction on $|E|$ that there is a colouring $c: E \to \mathbb{Z}$ such that $m_c(e) < f(e)$ for each edge e; the result will then follow from Corollary 1.1. We may suppose w.l.o.g. that G has no isolated vertices.

If $|U| \leqslant |W|$, let M be the matching whose existence was proved in Lemma 3.1. By the induction hypotheses, $G \backslash M$ has a colouring $c': E \backslash M \to \mathbb{Z}$ such that $m_{c'}(e) < f'(e)$ for each edge $e = uw$, where $f'(e) := \max\{d'(u), d'(w)\}$ and d' denotes degree in $G \backslash M$. Let $c(e) := c'(e)$ if $e \in E \backslash M$, and if $e \in M$ let $c(e)$ be greater than any colour in $c'(E \backslash M)$. Then $m_c(e) \leqslant m_{c'}(e) + 1$ for each edge $e = uw \notin M$, and if $m_c(e) = m_{c'}(e) + 1$ then $w \in V(M)$, which by Lemma 3.1 implies $u \in V(M)$, so that $f(e) = f'(e) + 1$. And if $e = uw \in M$ then clearly $m_c(e) < d(u)$. Thus $m_c(e) < f(e)$ for each edge e, as required.

If $|U| > |W|$ then Lemma 3.1 implies the existence of a nonempty matching M such that whenever $e = uw \in E$ and $u \in V(M)$, then $w \in V(M)$. Define c' and c as before, except that if $e \in M$ let $c(e)$ be less than any colour in $c'(E \backslash M)$. As before, it is easy to see that $m_c(e) < f(e)$ for each edge e, and so the proof of Theorem 3 is complete. ∎

3. LIST EDGE AND LIST TOTAL COLOURINGS OF MULTIGRAPHS

The remainder of this paper consists of a string of applications of Theorem 3. Our first application (Theorem 4) is to nonbipartite multigraphs. We shall need the following lemma, in which the case $\Delta(H) = 1$ (when H is a matching, and the conclusion implies $\chi'_{\text{list}}(G) \leqslant \Delta(G) + 1$) is perhaps of independent interest, since Conjecture A(a) implies that $\chi'_{\text{list}}(G) \leqslant \Delta(G) + 1$ for every simple graph G.

LEMMA 4.1. *If F, G, H are multigraphs, where F is bipartite and $G = F \cup H$, and*

$$f(e) := \max\{d_G(u) + d_H(w), d_H(u) + d_G(w)\}$$

for each edge $e = uw$, then G is edge-f-choosable.

Proof. We may clearly assume $E(F) \cap E(H) = \emptyset$. Since $d_H(v) \leqslant d_G(v)$ for each vertex v, it follows that $f(e) \geqslant d_H(u) + d_H(w)$ for each edge $e = uw$, and so we can easily colour the edges of H from their lists. If now $e = uw$ is an edge of F, then the number of different colours already used on edges adjacent to e is at most $d_H(u) + d_H(w)$, and so e effectively still has a list of size at least

$$f(e) - d_H(u) - d_H(w) = \max\{d_F(u), d_F(w)\}.$$

Thus the edges of F can be coloured by Theorem 3. ∎

THEOREM 4. *A multigraph G is edge-f-choosable, where*

$$f(e) := \max\{d(u), d(w)\} + \lfloor \tfrac{1}{2} \min\{d(u), d(w)\} \rfloor$$

for each edge $e = uw$. *In particular*, $\chi'_{\text{list}}(G) \leqslant \lfloor \tfrac{3}{2}\Delta(G) \rfloor$.

Proof. Let $\langle U, W \rangle$ be a largest cut of G, that is, a partition of $V(G)$ such that the bipartite spanning subgraph F comprising all edges between U and W has as many edges as possible. Then it is easy to see that $d_F(v) \geqslant \frac{1}{2} d_G(v)$ for each vertex v. (If not, then move v to the other side of the partition.) So if $H := G \backslash E(F)$ then $d_H(v) \leqslant \frac{1}{2} d_G(v)$ for each vertex v, and so

$$\begin{aligned} f(e) &= \max\{d_G(u) + \lfloor \tfrac{1}{2} d_G(w) \rfloor, d_G(w) + \lfloor \tfrac{1}{2} d_G(u) \rfloor\} \\ &\geqslant \max\{d_G(u) + d_H(w), d_H(u) + d_G(w)\} \end{aligned}$$

for each edge $e = uw$. The result follows from Lemma 4.1. ∎

The proof of Theorem 4 gives a polynomial-time algorithm for colouring the edges of a multigraph G with at most $\frac{3}{2}\Delta(G)$ colours. This is because the procedures in the proofs of Lemmas 1.1 and 3.1 are polynomial, and in the proof of Theorem 4 itself we do not really need a largest cut, but only one that satisfies the stated degree properties, which we can construct by a greedy algorithm.

With the aid of Theorems 3 and 4, we can obtain easy consequences for list total colourings from any result about list vertex colourings, such as the following result of Borodin [3, 4] and Erdös *et al.* [10].

THEOREM 5 [3, 4, 10]. *Let* $G = (V, E)$ *be a connected graph and let* $f\colon V \to \mathbb{N}$ *be a function such that* $f(v) \geqslant d(v)$ *for each vertex* v, *and at least one of the following holds*:

(a) $f(v) \geqslant d(v) + 1$ *for at least one vertex* v;

(b) *G has a block that is neither complete nor an odd cycle.*

Then G is (*vertex-*) *f-choosable.*

In applying Theorem 5 to a multigraph G, one would of course do better to replace $d(v)$ by $|N(v)|$, thereby effectively applying the theorem to the underlying simple graph of G. Theorems 3–5 then give the following results, which are proved by colouring the vertices first and then the edges. (See also Theorems C and D at the end of this paper.)

COROLLARY 5.1. *If* $G=(V, E)$ *is a bipartite multigraph and* $f\colon V \cup E \to \mathbb{N}$ *is a function that satisfies the hypotheses of Theorem 5 on each component of the underlying simple graph and is such that* $f(e) \geqslant \max\{d(u), d(w)\}+2$ *for each edge* $e=uw$, *then G is totally-f-choosable.*

COROLLARY 5.2. *If* $G=(V, E)$ *is an arbitrary multigraph and* $f\colon V \cup E \to \mathbb{N}$ *is a function that satisfies the hypotheses of Theorem 5 on each component of the underlying simple graph and is such that* $f(e) \geqslant \max\{d(u), d(w)\} + \lfloor \frac{1}{2} \min\{d(u), d(w)\} \rfloor + 2$ *for each edge* $e=uw$, *then G is totally-f-choosable. In particular,* $\chi''_{\text{list}}(G) \leqslant \lfloor \frac{3}{2}\Delta \rfloor + 2$ *for every multigraph with maximum degree* Δ.

4. TOTAL COLOURINGS OF MULTIGRPAHS

The proof [20–23] of Theorem A(b), that $\chi''(G) \leqslant \lfloor \frac{3}{2}\Delta \rfloor$ for any multigraph G with maximum degree $\Delta \geqslant 4$, is quite long and complicated. One can distinguish four cases, namely (in increasing order of length and difficulty) $\Delta=4$; $\Delta \geqslant 6$ and even; $\Delta \geqslant 7$ and odd; and (longest and hardest of all) $\Delta=5$. In view of this, it is perhaps of interest to note that Theorem 4 gives quite a short proof of the weaker result in Theorem 6. Note that Theorem 6 gives the best possible results for $\Delta=2$ and 3, the latter being due to Rosenfeld [24] and Vijayaditya [28].

THEOREM 6. *If G is a multigraph with maximum degree* $\Delta \geqslant 2$, *then* $\chi''(G) \leqslant \lfloor \frac{3}{2}\Delta \rfloor + 1$.

Proof. Choose G to be a counterexample with as few vertices as possible and, subject to this, as many edges as possible. Then G is 2-edge-connected and has a vertex s such that $d(v)=\Delta$ whenever $v \neq s$. If $d(s)=\Delta$ put $G^* := G$, otherwise form G^* by taking a copy G' of G, disjoint from G, and connecting s to its copy s' in G' by $\Delta - d(s)$ edges.

G^* is a Δ-regular multigraph with at most one cut-edge. By Tutte's characterization [27], G^* has a 2-factor T. Remove an edge from each odd cycle of T to form a spanning bipartite subgraph of G^* with maximum degree 2 and no component K_1 or K_2. Let F be a maximal spanning bipartite subgraph with these properties. If u, w are adjacent vertices of G^* such that $d_F(u)=d_F(w)=1$, then u, w are the endvertices of a path of even length in F (otherwise we could add the edge uw to F). So we can form a partial total colouring of G^* by colouring all edges of F, and all vertices of degree 1 in F, using only two colours.

Assume G^* is not K_4 or an odd cycle, which are easily dealt with separately. Since $\lfloor \frac{3}{2}\Delta \rfloor + 1 - 2 \geqslant \Delta$, with strict inequality if $\Delta \geqslant 4$, we can colour the remaining vertices of G^* at random with the remaining colours. Let $H := G^* \backslash E(F)$. To complete the colouring we must find a list edge colouring of H when each edge uw is given a list of size at least

$$\lfloor \tfrac{3}{2}\Delta \rfloor + 1 - 2 \geqslant (\Delta - 1) + \lfloor \tfrac{1}{2}(\Delta - 1) \rfloor \quad \text{if} \quad d_H(u) = \Delta - 1,\ d_H(w) = \Delta - 1,$$

$$\lfloor \tfrac{3}{2}\Delta \rfloor + 1 - 3 = (\Delta - 1) + \lfloor \tfrac{1}{2}(\Delta - 2) \rfloor \quad \text{if} \quad d_H(u) = \Delta - 1,\ d_H(w) = \Delta - 2,$$

$$\lfloor \tfrac{3}{2}\Delta \rfloor + 1 - 4 = (\Delta - 2) + \lfloor \tfrac{1}{2}(\Delta - 2) \rfloor \quad \text{if} \quad d_H(u) = \Delta - 2,\ d_H(w) = \Delta - 2.$$

Thus the colouring can be completed by Theorem 4. ▮

5. THE MAXIMUM AVERAGE DEGREE

The *maximum average degree* $\operatorname{mad}(G)$ of a multigraph G is the maximum value of $2\,|E(H)|/|V(H)|$ taken over all submultigraphs H of G. It is of interest because if G is a graph embedded in a surface then $\operatorname{mad}(G)$ is bounded above by a function of the characteristic of the surface (and, if desired, the girth of G). Indeed, several results about colourings were originally proved using $\operatorname{mad}(G)$. One example is Heawood's bound on the chromatic number of a surface, although the proof of this uses only the simple fact that a graph G is k-degenerate, where $k = \lfloor \operatorname{mad}(G) \rfloor$. More sophisticated uses of $\operatorname{mad}(G)$ to prove results on total colourings can be found in [9] and the last section of [7], and in Corollaries 7.1 and 7.2 below.

In order to state the main result of this section (Theorem 7), we need a definition. If $p \in \mathbb{N}$ define $q = q(p)$ and $r = r(p)$ by $p = \binom{q}{2} + r$ where $1 \leqslant r \leqslant q$, as in Table I.

For future reference, we note from Table 1 that

$$q - 1 \leqslant \tfrac{1}{2} p \qquad \text{with strict inequality if } p \notin \{2, 4\}, \tag{1}$$

and

$$\frac{1}{2}(q-3) + \frac{r}{q} \geqslant 0. \tag{2}$$

TABLE I

p	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	...
q	1	2	2	3	3	3	4	4	4	4	5	5	5	5	5	6	...
r	1	1	2	1	2	3	1	2	3	4	1	2	3	4	5	1	...

In order to simplify the proof of Theorem 7, it is convenient to allow $\Delta(G) \leqslant \Delta$ in its statement; but of course the interesting case is when $\Delta(G) = \Delta$. We conjecture that the annoying exceptions in part (b) of Theorem 7 are unnecessary.

THEOREM 7. *Let $k, \Delta \in \mathbb{N}$ with $\Delta \geqslant 2$, and let G be a multigraph with $\Delta(G) \leqslant \Delta$.*

(a) *If G is k-degenerate (in particular, if* $\mathrm{mad}(G) < k+1$) *then $\chi'_{\mathrm{list}}(G) \leqslant \Delta + k - 1$ and $\chi''_{\mathrm{list}}(G) \leqslant \Delta + k$.*

(b) *Suppose $\Delta - k \geqslant p \geqslant 2$ and* $\mathrm{mad}(G) \leqslant q + (r/q) + k - 1$, *where $q = q(p)$ and $r = r(p)$. Then $\chi'_{\mathrm{list}}(G) \leqslant \Delta + k - 1$. Moreover, if $p \notin \{2, 4\}$ and $(p, k) \neq (3, 1)$ or $(3, 2)$, or if $\Delta - k > p$, then $\chi''_{\mathrm{list}}(G) \leqslant \Delta + k$. In particular (taking $k = 1$), if*

$$\Delta(G) = \Delta$$

and

$\Delta \geqslant 3$	4	5	6	7	8	9	10	11	12	13	14	15	16	…
and														
$\mathrm{mad}(G) \leqslant 2\frac{1}{2}$	3	$3\frac{1}{3}$	$3\frac{2}{3}$	4	$4\frac{1}{4}$	$4\frac{1}{2}$	$4\frac{3}{4}$	5	$5\frac{1}{5}$	$5\frac{2}{5}$	$5\frac{3}{5}$	$5\frac{4}{5}$	6	…

then $\chi'_{\mathrm{list}}(G) = \Delta$, and if $\Delta \geqslant 6$, or $\Delta \geqslant 5$ and $\mathrm{mad}(G) \leqslant 3$, *or $\Delta \geqslant 4$ and* $\mathrm{mad}(G) \leqslant 2\frac{1}{2}$, *then $\chi''_{\mathrm{list}}(G) = \Delta + 1$.*

Proof. Define ε to be 0 or 1 according to whether we are proving the results for χ'_{list} or χ''_{list}; that is, according to whether we are colouring edges only, or edges and vertices. Let $G = (V, E)$ be a minimal counterexample to the theorem. We show first that if $e = uw \in E$ and $d(u) \leqslant \frac{1}{2}(\Delta + k - \varepsilon)$ then

$$d(w) \geqslant \Delta + k + 1 - d(u) > d(u). \tag{3}$$

For, suppose $d(w) \leqslant \Delta + k - d(u)$. We can colour $G - e$ by the minimality of G. If we are colouring edges only, then e now touches at most $\Delta + k - 2$ colours and so we can colour it from its list. If we are colouring vertices as well, then we first uncolour u, so that e touches at most $\Delta + k - 1$ colours and can be coloured from its list, and now u is adjacent or incident to at most $2d(u) < \Delta + k$ colours and it too can be coloured. This contradicts the choice of G as a counterexample and shows that (3) holds.

We next show that

$$\delta(G) \geqslant k + 1. \tag{4}$$

For, suppose $\delta(G) \leqslant k$. This is impossible if $k < \Delta$, since then $uw \in E$ and $d(u) \leqslant k$ imply $d(w) \geqslant \Delta + 1$ by (3). So we may suppose that $\Delta \leqslant k$. But it is easy to see that $\chi'_{\text{list}}(G) \leqslant 2\Delta - 1 \leqslant \Delta + k - 1$, and Corollary 5.2 gives $\chi''_{\text{list}}(G) \leqslant \lfloor \frac{3}{2}\Delta \rfloor + 2 \leqslant \Delta + k$ if $\Delta \geqslant 3$. It is an easy exercise to prove that $\chi''_{\text{list}}(G) \leqslant 4$ if $\Delta = 2$, and this contradiction proves (4). Now (a) immediately follows, since (4) cannot hold if G is k-degenerate.

Before proving (b), we introduce the concept of a *d-alternating subgraph.* This is a bipartite subgraph F of G with partite sets U, W such that if $u \in U$ and $w \in W$ then $d_F(u) = d_G(u) \leqslant d$ and

$$d_F(w) \geqslant d_G(w) + d - k - \Delta + 1. \tag{5}$$

We shall show that if $d \leqslant \frac{1}{2}(\Delta + k - \varepsilon)$ then

$$G \text{ contains no } d\text{-alternating subgraph.} \tag{6}$$

For, suppose it does. Colour all elements of $G - U$ from their lists, which is possible by the minimality of G. If $e = uw$ is an edge of F with $u \in U$, $w \in W$, then e now effectively has a list $L(e)$ of at least

$$|L(e)| \geqslant \Delta + k - 1 + d_F(w) - d_G(w) \geqslant d_F(w)$$

possible colours. Also $|L(e)| \geqslant d$ by (5). Thus $|L(e)| \geqslant \max\{d_F(u), d_F(w)\}$ for each edge $e = uw$ of F, and it follows from Theorem 3 that the edges of F can be coloured from their lists. Finally, if we are colouring vertices, then each vertex $u \in U$ is now adjacent or incident to at most $2d < \Delta + k$ colours, and so there is a colour in its list that we can give to it. This contradiction proves (6).

We now prove (b). If $W \subseteq V$ and $d \geqslant k + 1$, let $n_i(W) := |\{v \in W : d(v) = i\}|$,

$$l(d, W) := \sum_{i=k+1}^{d} n_i(W), \qquad h(d, W) := \sum_{i=\Delta+k+1-d}^{\Delta} (i + d - k - \Delta)\, n_i(W),$$

$n_i := n_i(V)$, $l(d) := l(d, V)$, and $h(d) := h(d, V)$. We shall show that if $k + 1 \leqslant d \leqslant \frac{1}{2}(\Delta + k - \varepsilon)$ then

$$l(d) = h(d) = 0 \qquad \text{or} \qquad l(d) < h(d). \tag{7}$$

For, let $X := \{v \in V : d(v) \leqslant d\}$ and $Y := \{v \in V : d(v) \geqslant \Delta + k + 1 - d\}$, so that $|X| = l(d)$ by (4). For $W \subseteq Y$, let $U(W) := \{x \in X : N(x) \subseteq W\}$, so that $U(Y) = X$ by (3). If $Y = \varnothing$ then $X = \varnothing$ and $l(d) = h(d) = 0$. So suppose $Y \neq \varnothing$ and suppose that $l(d) \geqslant h(d)$, that is, $|U(Y)| = |X| = l(d) \geqslant h(d) = h(d, Y)$. Choose $W \subseteq Y$ minimal such that $W \neq \varnothing$ and $|U(W)| \geqslant h(d, W)$, and let F be the subgraph of all edges between $U(W)$ and W. Then

(5) holds for each $w \in W$, since if $d_F(w) \leqslant d_G(w) + d - k - \Delta = h(d, W) - h(d, W \backslash \{w\})$ then

$$|U(W \backslash \{w\})| = |U(W)| - d_F(w) \geqslant h(d, W) - d_F(w) \geqslant h(d, W \backslash \{w\}),$$

contradicting the minimality of W. (Note that $N(U(W)) \subseteq W$ and $|N(U(W))| \geqslant k+1 \geqslant 2$, so that $W \backslash \{w\} \neq \emptyset$.) Since clearly $d_F(u) = d_G(u) \leqslant d$ for each $u \in U(W)$, F is a d-alternating subgraph, contrary to (6). This contradiction proves (7).

By (1), $q+k-1 \leqslant \frac{1}{2}p + k \leqslant \frac{1}{2}(\Delta + k)$, with strict inequality if $p \notin \{2, 4\}$ or $\Delta - k > p$, and so we may suppose that

$$k+1 \leqslant q+k-1 \leqslant \tfrac{1}{2}(\Delta + k - \varepsilon). \tag{8}$$

We now show that

$$l(q+k-1) < h(q+k-1). \tag{9}$$

For, if (9) does not hold, then (7) and (8) imply $l(q+k-1) = 0$, which by (4) and the definition of $l(d)$ implies $\delta(G) \geqslant q+k$. Since $\text{mad}(G) \leqslant q + (r/q) + k - 1$ and $r \leqslant q$, the only possibility is that G is $(q+k)$-regular and $r = q$, which implies $p = \binom{q+1}{2}$. Suppose first that $q \geqslant 3$. Then $p \geqslant \frac{3}{2}(q+1)$, and so $\Delta + k \geqslant 2k + p \geqslant \frac{3}{2}(k+q) + 2$. Since G is $(k+q)$-regular, Theorem 4 implies $\chi'_{\text{list}}(G) \leqslant \lfloor \frac{3}{2}(k+q) \rfloor \leqslant \Delta + k - 2$ and Corollary 5.2 implies $\chi''_{\text{list}}(G) \leqslant \lfloor \frac{3}{2}(k+q) \rfloor + 2 \leqslant \Delta + k$. This is a contradiction. Thus we must have $q = 2$, so that $\Delta \geqslant k + p = k + 3$. Then Theorem 4 implies $\chi'_{\text{list}}(G) \leqslant \lfloor \frac{3}{2}(k+2) \rfloor \leqslant 2k + 2 \leqslant \Delta + k - 1$, and Corollary 5.2 implies $\chi''_{\text{list}}(G) \leqslant \lfloor \frac{3}{2}(k+2) \rfloor + 2 \leqslant 2k + 3 \leqslant \Delta + k$ if $k \geqslant 3$. This contradiction proves (9).

Now, if $\text{mad}(G) \leqslant a$ then $\sum i n_i \leqslant a \sum n_i$ and so

$$\sum_{i<a} (a-i)\, n_i \geqslant \sum_{i>a} (i-a)\, n_i.$$

Putting $a = q + (r/q) + k - 1$ from the statement of the theorem gives

$$\begin{aligned}\sum_{i=k+1}^{q+k-1} \left(q + \frac{r}{q} + k - 1 - i\right) n_i &\geqslant \sum_{i=q+k}^{\Delta} \left(i - q - \frac{r}{q} - k + 1\right) n_i \\ &\geqslant \sum_{i=0}^{p-q} \left(p - i - q - \frac{r}{q} + 1\right) n_{\Delta - i}\end{aligned} \tag{10}$$

(replacing i by $\Delta - i$), since $\Delta - k \geqslant p$. But we can rearrange the LHS of (10) and then use (7), (8), and (9) to give

$$\sum_{i=k+1}^{q+k-1}\left(q+\frac{r}{q}+k-1-i\right)n_i$$
$$=\left(\sum_{d=k+1}^{q+k-2} l(d)\right)+\frac{r}{q}\,l(q+k-1)$$
$$<\left(\sum_{d=k+1}^{q+k-2} h(d)\right)+\frac{r}{q}\,h(q+k-1)$$
$$=\left(\sum_{d=k+1}^{q+k-2}\ \sum_{i=\Delta+k+1-d}^{\Delta}(i+d-k-\Delta)\,n_i\right)$$
$$+\frac{r}{q}\sum_{i=\Delta+2-q}^{\Delta}(i+q-1-\Delta)\,n_i$$
$$=\sum_{i=\Delta+2-q}^{\Delta}\left(\left(\sum_{d=\Delta+k+1-i}^{q+k-2}(i+d-k-\Delta)\right)+\frac{r}{q}(i+q-1-\Delta)\right)n_i$$
$$=\sum_{i=0}^{q-2}\left(\left(\sum_{d=k+1+i}^{q+k-2}(d-k-i)\right)+\frac{r}{q}(q-1-i)\right)n_{\Delta-i}$$
$$=\sum_{i=0}^{q-2}\left(\left(\sum_{j=1}^{q-i-2} j\right)+\frac{r}{q}(q-1-i)\right)n_{\Delta-i}$$
$$=\sum_{i=0}^{q-2}\left(\frac{1}{2}(q-i-1)(q-i-2)+r-\frac{r}{q}(1+i)\right)n_{\Delta-i}$$
$$=\sum_{i=0}^{q-2}\left(\frac{1}{2}q(q-1)+r-i-q-\frac{r}{q}+1-i\left(q-\frac{5}{2}-\frac{1}{2}i+\frac{r}{q}\right)\right)n_{\Delta-i}$$
$$\leqslant\sum_{i=0}^{q-2}\left(p-i-q-\frac{r}{q}+1\right)n_{\Delta-i} \tag{11}$$

since $p=\frac{1}{2}q(q-1)+r$ and when $i\leqslant q-2$ then $q-\frac{5}{2}-\frac{1}{2}i+(r/q)\geqslant\frac{1}{2}q-\frac{3}{2}+(r/q)\geqslant 0$ by (2). Now (11) contradicts (10) because $q-2\leqslant p-q$ by (1). This completes the proof of Theorem 7. ∎

Chen and Wu [9] proved that a planar graph with girth g and maximum degree Δ satisfies $\chi''(G)=\Delta+1$ if $\Delta\geqslant 8$ and $g\geqslant 4$, or $\Delta\geqslant 6$ and $g\geqslant 5$, or $\Delta\geqslant 4$ and $g\geqslant 8$. It follows from Corollary 7.1 below that the first two of these conditions actually imply $\chi''_{\text{list}}(G)=\Delta+1$. In [7] we improved some of Chen and Wu's results by proving $\chi''(G)=\Delta+1$ if $\Delta\geqslant 5$ and $g\geqslant 5$, or $\Delta\geqslant 4$ and $g\geqslant 6$, or $\Delta\geqslant 3$ and $g\geqslant 10$, and we would conjecture that these conditions suffice to ensure $\chi''_{\text{list}}(G)=\Delta+1$; but we have not been able to prove this.

Corollary 7.1. *If G is a graph with girth g and maximum degree Δ that is embedded in a surface of nonnegative characteristic* (*the plane, projective plane, torus, or Klein bottle*), *and if*

$$g \geqslant 3 \quad g \geqslant 4 \quad g \geqslant 5 \quad g \geqslant 6 \quad \text{or} \quad g \geqslant 10,$$

then

$$\chi'_{\text{list}}(G) = \Delta \quad \text{if} \quad \Delta \geqslant 16 \quad \Delta \geqslant 7 \quad \Delta \geqslant 5 \quad \Delta \geqslant 4 \quad \text{or} \quad \Delta \geqslant 3$$

and

$$\chi''_{\text{list}}(G) = \Delta + 1 \quad \text{if} \quad \Delta \geqslant 16 \quad \Delta \geqslant 7 \quad \Delta \geqslant 6 \quad \Delta \geqslant 5 \quad \text{or} \quad \Delta \geqslant 4.$$

Proof. Let $H \subseteq G$, say $H = (V, E, F)$. Euler's formula $|V| - |E| + |F| \geqslant 0$ can be rewritten in the form

$$2(g-2)|E| - 2g|V| + 4|E| - 2g|F| \leqslant 0,$$

which implies $2|E|/|V| \leqslant 2g/(g-2)$ since $4|E| \geqslant 2g|F|$ for a graph with girth g. This in turn implies that $\text{mad}(G) \leqslant 2g/(g-2)$.

Thus

$$g \geqslant 3 \quad 4 \quad 5 \quad 6 \quad 10$$

implies

$$\text{mad}(G) \leqslant 6 \quad 4 \quad 3\tfrac{1}{3} \quad 3 \quad 2\tfrac{1}{2}.$$

The results can now be read from Theorem 7. ∎

We shall improve the result for $g = 3$ in the next section. For other values of g, as already remarked, the known conditions [7] for $\chi''(G)$ to equal $\Delta(G) + 1$ agree with those given for $\chi'_{\text{list}}(G)$ to equal $\Delta(G)$ in Corollary 7.1, and we know of no conditions for $\chi'(G)$ to equal $\Delta(G)$ that are weaker than these. (However, see the next section.)

For an integer $\chi \leqslant 0$, let

$$T(\chi) := \lceil 2 + \sqrt{16 - 6\chi} \rceil \qquad \text{and} \qquad H(\chi) := \lceil \tfrac{1}{2}(5 + \sqrt{49 - 24\chi}) \rceil.$$

(The definition of $H(\chi)$ is the same as that of the Heawood number $\lfloor \frac{1}{2}(7 + \sqrt{49 - 24\chi}) \rfloor$ except when $49 - 24\chi$ is a perfect square.) It is easy to check that $H - 1 \leqslant T \leqslant H$.

Corollary 7.2. *If G is a simple graph with maximum degree Δ embedded in a surface of characteristic $\chi \leqslant 0$, and $\Delta \geqslant \binom{t}{2} + 1$ where $t := T(\chi)$, then $\chi'_{\text{list}}(G) = \Delta$ and $\chi''_{\text{list}}(G) = \Delta + 1$.*

Proof. It suffices to prove that $\operatorname{mad}(G) \leqslant t$, since then we can apply Theorem 7 with $k=1$, $p=\binom{t}{2}$, and $q=r=t-1$.

Let $H \subseteq G$, say $H=(V, E, F)$, and let the average degree of H be a, so that $a|V|=2|E| \geqslant 3|F|$ and a standard application of Euler's formula $|V|-|E|+|F| \geqslant \chi$ gives

$$(a-6)|V| = 2|E| - 6|V| \leqslant -6\chi - 4|E| + 6|F| \leqslant -6\chi. \tag{12}$$

If $|V| \leqslant t+1$ then certainly $a \leqslant \Delta \leqslant t$. So we may suppose $|V| \geqslant t+2$. If $a > t$ then (12) gives $(t-6)(t+2) < -6\chi$, whence $t < 2+\sqrt{16-6\chi}$, and this contradiction completes the proof. ∎

Note that if we had used only $|V| \geqslant t+1$ in the above proof, then we would have got the marginally weaker result with $t=H(\chi)$ instead of $t=T(\chi)$. For most values of χ, the result can be improved if, instead of rounding up to the next integer in the definition of $T(\chi)$, we instead round up to the next number of the form $q+(r/q)$ where $1 \leqslant r \leqslant q$; but the result that one obtains in this way is messy to state. For graphs of girth $g \geqslant 4$, the result can also be improved by taking note of the girth, as we did in Corollary 7.1; we omit the details.

6. PLANAR GRAPHS OF LARGE MAXIMUM DEGREE

For a simple planar graph G with maximum degree Δ we have the result of Vizing [30] that $\chi'(G)=\Delta$ if $\Delta \geqslant 8$ (conjectured to hold if $\Delta \geqslant 6$); the result of Borodin [6] that $\chi'_{\text{list}}(G)=\Delta$ if $\Delta \geqslant 14$; and the result of Borodin [5] that $\chi''(G)=\Delta+1$ if $\Delta \geqslant 14$, which we have extended in [8] to $\Delta \geqslant 11$. This concluding section of our paper is devoted to the proof that $\chi'_{\text{list}}(G)=\Delta$ and $\chi''_{\text{list}}(G)=\Delta+1$ if $\Delta \geqslant 12$; this improves on the result of [6] and on the result for $g=3$ in Corollary 7.1; the proof is based on ideas from [5].

Before proving this result, we need some definitions. A *k-vertex* is a vertex with degree k. A *k-face* is a face with k edges. A 2-*alternating cycle* in a graph G is a cycle of even length in which alternate vertices have degree 2 in G. This important concept, introduced in [5], was the precursor of the d-alternating subgraphs used in the proof of Theorem 7. Unfortunately, the absence of 2- and 3-alternating subgraphs is not enough to ensure the truth of the next theorem, and so we need to introduce something more general than the latter. Accordingly, we define a 3-*alternator* to be a bipartite subgraph F of G with partite sets U, W such that, for each $u \in U$, $2 \leqslant d_F(u) = d_G(u) \leqslant 3$, and for each $w \in W$, either $d_F(w) \geqslant 3$ or w has exactly two neighbours in U, both with degree exactly $14-d_G(w)$ (this last being possible only if $d_G(w)=11$ or 12).

We shall need the following result, which seems sufficiently important to be called a theorem. The figure 13 here is best possible, although our construction, proving this, is too complicated to show here. (If only 2-alternating cycles are forbidden, then 15 is the best possible, as shown in [4].)

THEOREM 8. *Let H be a simple graph embedded in a surface of non-negative characteristic, containing no 2-alternating cycle or 3-alternator, such that $\delta(H) \geqslant 2$. Then H contains an edge $e = uw$ such that $d_H(u) + d_H(w) \leqslant 13$.*

Proof. It is convenient to prove the result for the wider class of graphs in which we allow H to have loops and parallel edges, provided that no face-boundary consists solely of a loop or a pair of parallel edges, and no vertex of degree 2 separates two triangular faces. Let $H = (V, E, F)$ be a counterexample to this more general result with as few vertices as possible and, subject to this, as many edges as possible. Note that each face of F has a connected boundary, since each component of the boundary must contain a vertex with degree at least 7 in H (as in the proof of the claim below), and we could add an edge joining two such vertices without violating any of the hypotheses of the theorem. In particular, H is connected. Let H^* be the graph obtained by deleting all 2-vertices from H along with their incident edges.

CLAIM. *H^* is a triangulation.*

Proof of the Claim. The neighbours u, w of a 2-vertex v in H both have degree at least 12, and so v must lie in the boundary of a 3-face of H, since otherwise we could join u, w by an edge passing close to v without violating any of the hypotheses of the theorem. (Note that $u \neq w$, since otherwise we have a 2-alternating 2-cycle.) Suppose now that there is a nontriangular face f in H^*. Let x, y, z be three consecutive vertices in the boundary of f, chosen so that $d_H(y)$ is as small as possible. If $d_H(y) \leqslant 6$ then $d_H(x) \geqslant 8$ and $d_H(z) \geqslant 8$, since $d_H(u) + d_H(w) \geqslant 14$ for each edge uw; otherwise, $d_H(x) \geqslant d_H(y) \geqslant 7$ and $d_H(z) \geqslant 7$. In either case we could add the edge xz inside f without violating any of the hypotheses of the theorem, and this contradiction proves the claim.

Let U be the set of vertices of H with degree at most 3 and let $W := N(U)$. If $X \subseteq E(H)$, write $U(X)$, $W(X)$ for the vertices in U, W that are incident with edges in X. Let X be a maximal subset of the edges between U and W such that

(a) for each $u \in U(X)$, $d_X(u) = 1$,

(b) for each $w \in W(X)$, $d_X(w) \leqslant 2$ and w has at most one X-neighbour in U of degree $14 - d_G(w)$, and

(c) $N(U \setminus U(X)) \subseteq W \setminus W(X)$.

Because of the absence of 3-alternators, if $U\backslash U(X)\neq\emptyset$ then there exists a $w\in N(U\backslash U(X))$ such that if we add to X the edges between w and $U\backslash U(X)$, then the resulting set satisfies (a)–(c) and violates the maximality of X. Thus $U(X)=U$. If $uw\in X$ ($u\in U$, $w\in W$) then we call w the *master* of u and u a *dependent* of w.

We now use the method of redistribution of charge in order to obtain a contradiction. We assign a "charge" $M(x)$ to each element $x\in V\cup F$, where

$$M(x):=\begin{cases} d(x)-6 & \text{if} \quad x\in V,\\ 2r(x)-6 & \text{if} \quad x\in F,\end{cases}$$

where $r(f)$ denotes the number of edges around face f. Euler's formula $|V|-|E|+|F|\geqslant 0$ can be rewritten in the form $(2|E|-6|V|)+(4|E|-6|F|)\leqslant 0$, which implies that

$$\sum_{x\in X\cup F} M(x)=\sum_{v\in V}(d(v)-6)+\sum_{f\in F}(2r(f)-6)\leqslant 0. \tag{13}$$

We shall now redistribute the charge, without changing its sum, in such a way that the sum is provably positive, and this contradiction will prove the theorem. Note that, by the claim and the absence of 2-alternating cycles, each 2-vertex lies between a 3-face and a k-face ($k=4$ or 5). The rules for redistribution of charge are as follows:

R1: Each 2-vertex receives charge 2 from its 4-face or 5-face, and also 2 from its master vertex.

R2: Each 3-vertex receives 2 from its master vertex and $\frac{1}{2}$ from each of its other two neighbours.

R3: Each 4-vertex or 5-vertex receives $\frac{1}{2}$ from each of its neighbours.

Let $M^*(x)$ be the resulting charge on x, so that $\sum_{x\in V\cup F} M^*(x)\leqslant 0$ by (13).

It is easy to see that $M^*(f)=0$ for each face f: a 3-face still has the zero charge it started with, a 4-face started with charge 2 and has given it to its unique 2-vertex, while a 5-face started with charge 4 and has given it equally to its two 2-vertices.

We now prove that $M^*(v)\geqslant 0$ for each vertex v, with strict inequality if $d(v)\notin\{2,3,4,6,12\}$. If v has given $\frac{1}{2}$ to a neighbour u, then we call u a **-neighbour* of v; clearly then $d(u)\leqslant 5$ and so $d(v)\geqslant 9$. Thus we can deal with the cases $d(v)\leqslant 8$ as follows: $M^*(v)=M(v)+4=0$ if $d(v)=2$; $M^*(v)=M(v)+3=0$ if $d(v)=3$; $M^*(v)=M(v)+2=0$ if $d(v)=4$; $M^*(v)=M(v)+2.5=1.5$ if $d(v)=5$; $M^*(v)=M(v)=0$ if $d(v)=6$; and $M^*(v)=M(v)>0$ if $d(v)=7$ or 8. If $d(v)=9$ or 10 then v can give $\frac{1}{2}$ to at most $\frac{1}{2}d(v)$ *-neighbours and so $M^*(v)\geqslant M(v)-\frac{1}{4}d(v)=d(v)-6-\frac{1}{4}d(v)>0$. If

$d(v) = 11$ then v can give at most 2 to a dependent of degree 3 and $\frac{1}{2}$ to 4 *-neighbours, so that $M^*(v) \geqslant 11 - 6 - 2 - 2 > 0$. If $d(v) = 12$ then v can give 2 to two dependents, but at most one of these can have degree 2, and so the number of dependents plus *-neighbours cannot exceed 6; hence $M(v) \geqslant 12 - 6 - 2 \cdot 2 - 4 \cdot \frac{1}{2} = 0$. Finally, if $d(v) \geqslant 13$ then the number of dependents plus *-neighbours can be at most $\frac{1}{2} d(v) + 1$ (with equality only if there are two dependent 2-vertices); thus $M^*(v) \geqslant d(v) - 6 - 2 \cdot 2 - \frac{1}{2}(\frac{1}{2} d(v) - 1) > 0$.

Since $\sum_{v \in V} M^*(v) \leqslant 0$, it follows that $M^*(v) = 0$ and $d(v) \in \{2, 3, 4, 6, 12\}$ for each vertex v. Moreover, each vertex of degree 12 has two dependents, four *-neighbours, and (therefore, by the claim) six neighbours of degree 12. It follows that there are no 2-vertices, since no 12-vertex is adjacent to a nondependent 2-vertex. Moreover, the 12-vertices induce a 6-regular subgraph of H. which must be a triangulation by Euler's formula. Hence there are no 4-vertices or 6-vertices either, and every 12-vertex is adjacent to six 3-vertices. But this contradicts the hypothesis that there is no 3-alternator, and this contradiction completes the proof of Theorem 8. ▮

We can now prove our final theorem.

THEOREM 9. *Let $\Delta \geqslant 12$ and let G be a simple graph with maximum degree $\Delta(G) \leqslant \Delta$, embedded in a surface of nonnegative characteristic. Then $\chi'_{\text{list}}(G) \leqslant \Delta$ and $\chi''_{\text{list}}(G) \leqslant \Delta + 1$. In particular, if $\Delta(G) = \Delta$, then $\chi'_{\text{list}}(G) = \Delta$ and $\chi''_{\text{list}}(G) = \Delta + 1$.*

Proof. Let G be a minimal counterexample to Theorem 9 (with $\Delta(G) \leqslant \Delta$). Suppose first that G contains an edge $e = uw$ with $d(u) + d(w) \leqslant 13$. Without loss of generality $d(u) \leqslant d(w)$, so that $d(u) \leqslant 6$. Colour all edges and (if appropriate) vertices of $G - e$ from their lists. If we are colouring vertices, erase the colour of u. There are now at least $\Delta - 11 \geqslant 1$ colours available to give to e, so colour e with one of them. If we are colouring vertices, then there are now at least $\Delta + 1 - 12 \geqslant 1$ colours available for u. Thus we can colour all elements of G.

This contradiction shows that in fact $d(u) + d(w) \geqslant 14$ for every edge $e = uw$ of G. Thus G does not satisfy the conclusion of Theorem 8, and we deduce that it cannot satisfy the hypotheses either. It is easy to see that $\delta(G) \geqslant 2$, and so G must contain a 2-alternating cycle or a 3-alternator.

Suppose first that G contains a 2-alternating cycle C. Remove the edges and 2-vertices of C from G, and colour the remaining edges and (if appropriate) vertices of G from their lists, which is possible by the minimality of G as a counterexample. There are now at least two colours available for each edge of C, and so these edges can be coloured by Theorem 3; and now (if we are colouring vertices) the vertices of C are easily coloured. Thus G is not a counterexample, which is a contradiction.

Hence G must contain a 3-alternator F. Remove from G the edges, 2-vertices, and 3-vertices of G that are in F, and colour the remaining edges and (if appropriate) vertices of G from their lists. The number of colours available for an edge $e=uw$ of F is now at least $d_F(w)\geqslant 3\geqslant d(u)$ if $d_G(w)\geqslant 12$, unless $d_G(w)=12$ and $d_F(w)=2$ in which case $d(u)=2$; and at least $d_F(w)+\Delta-d_G(w)\geqslant 2+1=3\geqslant d(u)$ if $d_G(w)=11$. Thus the edges of F can be coloured by Theorem 3. Again, the vertices are easily coloured (if we are colouring vertices), and this contradiction completes the proof of Theorem 9. ▌

7. TWO CONJECTURES

We conclude the paper with two conjectures about $\chi''_{\rm list}(G)$. Vizing [32] and Erdős *et al.* [10] proved the analogue of Brooks's theorem for list colourings, that $\chi_{\rm list}(G)\leqslant\Delta$ for connected G unless G is a complete graph or an odd cycle. Thus Galvin's theorem immediately implies the following result, which also follows from Corollary 5.1.

THEOREM C. *If $G=(V,E)$ is a bipartite multigraph with maximum degree $\Delta\geqslant 2$, then $\chi''_{\rm list}(G)\leqslant\Delta+2$. In fact, if $f(v)=\Delta$ for each $v\in V$ and $f(e)=\Delta+2$ for each $e\in E$, then G is totally-f-choosable.*

The complete-bipartite graphs show that there is no constant c such that a multigraph G is totally-f-choosable whenever $f(e)=\Delta$ for each $e\in E$ and $f(v)=\Delta+c$ for each $v\in V$; and also that if $f(x)=\Delta+1$ for each $x\in V\cup E$, then G need not be totally-f-choosable. The obvious remaining question is, if $f(e)=\Delta+1$ for each $e\in E$, how big does each $f(v)$ need to be to ensure total-f-choosability?

Conjecture C. If $G=(V,E)$ is bipartite and $f(e)=\Delta+1$ for each $e\in E$ and $f(v)=\Delta+2$ for each $v\in V$, then G is totally-f-choosable.

For nonbipartite graphs, as we have already observed in Corollary 5.2, our list analogue of Shannon's theorem (Theorem 4 in Section 3) implies immediately:

THEOREM D. *If $G=(V,E)$ is a multigraph with maximum degree Δ, then $\chi''_{\rm list}(G)\leqslant\lfloor\frac{3}{2}\Delta\rfloor+2$. In fact, if $f(v)=\Delta$ for each $v\in V$ and $f(e)=\lfloor\frac{3}{2}\Delta\rfloor+2$ for each $e\in E$, and G is connected and not a complete graph or an odd cycle, then G is totally-f-choosable.*

Conjecture D. $\chi''_{\rm list}(G)\leqslant\lfloor\frac{3}{2}\Delta\rfloor$ for every multigraph G with maximum degree $\Delta\geqslant 4$. Moreover, if $f(v)=\Delta$ for each $v\in V$ and $f(e)=\lfloor\frac{3}{2}\Delta\rfloor$ for each $e\in E$, and G is connected and not complete, then G is totally-f-choosable.

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