The 7/5-Conjecture Strengthens Itself

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ABSTRACT

The following is proved: if every bridgeless graph G has a cycle cover of length at most $7/5|E(G)|$, then every bridgeless graph G has a cycle cover of length at most $7/5|E(G)|$ such that any edge of G is covered once or twice. © 1995 John Wiley & Sons, Inc.

1. INTRODUCTION

A *cycle cover* C of a graph $G = (V, E)$ is a collection of cycles $\{C_1, \dots, C_t\}$ covering E. Its *length* is $l(C) = |C_1| + \cdots + |C_t|$. The *cycle covering ratio* $r(G)$ of graph $G = (V, E)$ with no cutedges is defined as the minimum of $l(C)/|E|$ over all the cycle coverings C of G. It was remarked [1,2,3] that there were many 2-edge-connected graphs with covering ratio equal to 7/5. There exists a folklore conjecture that

$$r(G) \leq 7/5 \quad \text{for any 2-edge-connected graph } G.$$

Call a cycle covering C of G *good* if any edge of G is covered by at most two cycles. The well-known *Cycle Double Cover (CDC) Conjecture* is equivalent to the statement that any 2-edge-connected graph has a good covering. U. Jamshy and M. Tarsi [4] showed that the validity of the 7/5-conjecture above implies the CDC-conjecture. Here we use their approach to show the following:

Theorem. The 7/5-conjecture implies that for any 2-edge-connected graph G there exists a good cycle covering of G of length at most $7/5|E(G)|$.

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2. PROOF OF THE THEOREM

Let $c(G)$ (respectively, $p(G)$) denote the length of the shortest cycle covering (respectively, the length of the shortest postman tour) of G. The following fact is obvious.

Observation. If $c(G) \leq p(G) + 1$, then any cycle covering of $E(G)$ of length $c(G)$ is a good covering. ∎

The observation may be applied to the Peterson graph P since $c(P) = 21 = 1 + p(P)$. Denote by kP the result of replacing every edge of P by a path of length k. Let $e = (x,y)$ be an edge in kP. Denote by R_k the graph obtained from kP by deleting e and adding two vertices x' and y' and two edges (x,x') and (y,y'). Then R_k has $15k - 3$ vertices and $15k + 1$ edges.

Now, let G be an arbitrary 2-edge-connected graph and $k = 2|E(G)|$. Our aim is to prove that there exists a good cycle covering of G of length at most $7k/10$ provided the 7/5-conjecture is true. Construct the graph H from G by replacing each edge (a,b) of G by a copy $R_k(a,b)$ of R_k so that x' coincides with a and y' coincides with b. Then H has $|V(G)| + (15k - 5)|E(G)|$ vertices and $|E(G)|(15k + 1) = k(15k + 1)/2$ edges. By the 7/5-conjecture, there is a cycle covering $C = \{C_1, \ldots, C_t\}$ of H of length at most $7k(15k + 1)/10$. Remark that for any copy R_k the edges (x,x') and (y,y') belong to the same cycles of C. This implies that

(1) C induces a cycle covering $C(a,b)$ on the copy of kP from which $R_k(a,b)$ was obtained, and
(2) C induces a cycle covering $C(G) = \{C'_1, \ldots, C'_t\}$ of G in the following way: $(a,b) \in C'_i$ if and only if the edge (x',x) in $R_k(a,b)$ belongs to C_i.

Since $c(P) = 21$, the length of $C(a,b)$ is at least $21k$, and, in view of the Observation, if $C(a,b)$ is not a good covering then its length is at least $22k$. Hence assuming that C is not good, we deduce that

$$\begin{aligned} l(C) &\geq 21k(|E(G)| - 1) + 22k = 21k(k/2 - 1 + 22/21) \\ &= 10.5k(k + 2/21) > (7/5)k(15k + 1)/2 = (7/5)|E(H)|, \end{aligned}$$

a contradiction. Thus, C is a good covering. For every copy $R_k(a,b)$, we may consider the edge (x,x') as belonging to G and the edge (y,y') as the edge completing $R_k(a,b)$ to R_k. As remarked, $l(C(a,b)) \geq 21k$ for each $(a,b) \in E(G)$ and so, to cover all the edges of H except the edges of the kind (x,x') we spend at least $21k^2/2$ edges of cycles of C. Hence the length of the covering $C(G)$ is equal to

$$l(C) - \sum_{(a,b)\in E(G)} l(C(a,b)) \le 7k(15k + 1)/10 - 21k^2/2$$
$$\le 7/10k = 7/5|E(G)|$$

and this covering is good, as C is good. ∎

3. DISCUSSION

Conjecture. For any 2-edge-connected graph G there exists a good cycle covering of G of length at most $|E(G)| + |V(G)| - 1$.

It can be shown that the minimal counterexample to the conjecture is 3-connected and has no 3- and 4-cycles.

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