A Characterization of Seymour Graphs

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ABSTRACT

A connected undirected graph G is called a *Seymour graph* if the maximum number of edge disjoint T*-cuts* is equal to the cardinality of a minimum T*-join* for every even subset T of $V(G)$. Several families of graphs have been shown to be subfamilies of Seymour graphs (Seymour *J. Comb. Theory B* **49** (1990), 189–222; *Proc. London Math. Soc. Ser. (3)* **42** (1981), 178–192; Gerards, *J. Comb. Theory B* **55** (1992), 73–82; Szigeti, (1993).) In this paper we prove a characterization of Seymour graphs which was conjectured by Sebö and implies the results mentioned above.

1. INTRODUCTION

Graphs in this paper are undirected, connected, and may have loops and multiple edges. Let G be a graph. For $F \subseteq E(G)$ and $x, y \in V(G)$, we write $xy \in F$ if some edge of G with endpoints x and y is in F.

For $X \subseteq V(G)$, the *cut* $\delta(X)$ is the set of edges connecting X and $V(G) \setminus X, N(X) = \{v \in V(G) \setminus X : v \text{ has neighbors in } X\}$. If $X = \{x\}$ we write $\delta(x), N(x)$. For $F \subseteq E(G)$

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and $v \in V(G)$, by $d_F(v)$ we denote the number of edges in F incident with v, each loop being counted twice. A pair (G, T) where T is an even subset of $V(G)$ is called a *graft.*

Let (G, T) be a graft. If $|X \cap T|$ is odd the cut $\delta(X)$ is called a T-*cut.* A set F of edges of G is a T-*join* if $T = \{v \in V(G) | d_F(v) \text{ is odd}\}$. Let $\nu(G, T)$ and $\tau(G, T)$ denote the maximum number of edge disjoint T-cuts and the cardinality of a minimum T-join in G, respectively.

It is easy to see that each T-join has at least one edge in common with each T-*cut.* Therefore,

$$\nu(G, T) \leq \tau(G, T). \tag{1}$$

In general, we do not have equality in (1), $(K_4, V(K_4))$ is an example. However, Seymour showed that bipartite [6] and series-parallel graphs [5] satisfy (1) with equality for every even vertex subset T.

Motivated by these results we call a graph G a *Seymour graph* if (1) holds with equality for all even subsets $T \subseteq V(G)$.

Gerards [1] proved that the family of Seymour graphs includes all graphs containing neither an odd K_4 nor an odd prism. Note that this extends Seymour's results since no bipartite or series-parallel graph contains either an odd K_4 or an odd prism. Yet, some odd $K_4 - s$ and odd prisms are Seymour graphs. Recently Szigeti [7] refined the result of Gerards by showing that if a graph is not a Seymour graph then it contains either an odd K_4 or an odd prism which is not a Seymour graph. Sebö (unpublished) conjectured a necessary and sufficient condition for a graph to be a Seymour graph. His conjecture is stated in terms of conservative weightings and implies the result of Szigeti (and whence other results mentioned above).

Let G be a graph and $\mathbf{w}$ be a ± 1 valued weighting defined on edges of G. The weighting $\mathbf{w}$ is called *conservative* if G has no cycle of negative total weight (a loop is considered as a cycle). For any $F \subseteq E(G)$, the weighting $\mathbf{w}_F$ on $E(G)$ is defined by the equation:

$$\mathbf{w}_F(e) := \begin{cases} -1 & \text{if } e \in F, \\ +1 & \text{if } e \notin F. \end{cases}$$

The following observation (Guan's lemma [4]) reveals a one-to-one correspondence between conservative weightings and T-joins of minimum cardinality:

A T-join F is of minimum cardinality if and only if the weighting $\mathbf{w}_F$ is conservative.

A *conservative graph* is a pair $(G, \mathbf{w})$ consisting of a graph G and a conservative weighting $\mathbf{w}$. Given a conservative graph $(G, \mathbf{w})$, we denote $E^-(\mathbf{w}) = \{e \in E(G) : \mathbf{w}(e) = -1\}$ and $T(\mathbf{w}) = \{v \in V(G) : d_{E^-(\mathbf{w})}(v) \text{ is odd}\}$. In the following we shall assume that $E^-(\mathbf{w}) \neq \emptyset$.

Given a conservative graph $(G, \mathbf{w})$, a cycle C of G is called a $\mathbf{w}$-*zero cycle* if the total weight of the edges of C is equal to zero. A graph G is an *odd* K_4 if it is a subdivision of K_4 such that each cycle bounding a face of G has an odd length. A graph G is an *odd prism* if it is a subdivision of triangular prism such that each cycle bounding a triangular face of G has an odd length while each cycle bounding a quadrangular face has an even length.

Conjecture (Sebö). A graph G is not a Seymour graph if and only if there exist a conservative weighting $\mathbf{w}$ and $\mathbf{w}$-zero cycles C_1, C_2 such that the graph $C_1 \cup C_2$ is either an odd K_4 or an odd prism.

By Guan's lemma the problem of deciding if an edge weighting is conservative is equivalent to the problem of deciding if a T-join is minimum. Since even the weighted version of the latter problem is polynomially solvable [2, p. 241], the conjecture implies that the class of Seymour graphs belongs to co-NP. The 'if' part of the conjecture was shown to be true by Sebö (unpublished).

In Section 3 we give a complete proof of the conjecture modulo Lemma 1. Section 2 contains basic known results needed in the proof. Lemma 1 is proved in Sections 4 and 5.

2. BACKGROUND

This section presents several known results to be of use further. To state them some extra notation and definitions are needed.

A connected graph G is called *1-extendable* if every edge of G lies in a perfect matching. A graph G is *factor-critical* if $G-x$ has a perfect matching for each $x \in V(G)$ and *bicritical* if $|E(G)| \geq 1$ and $G-\{x,y\}$ has a perfect matching for each pair of vertices $x,y \in V(G)$. A subdivision of a graph G is said to be even if the number of new vertices inserted in every edge of G is even. Clearly, any even subdivision of K_4 (respectively, of triangular prism) is an odd K_4 (respectively, an odd prism).

The first result is an easy consequence of Theorem 5.4.11 in [2].

Theorem (Lovász). *Let G be a 1-extendable non-bipartite graph. Then G contains an even subdivision of either K_4 or a triangular prism.* ∎

Recently Lucchesi and Carvalho [3] provided a simple proof of Theorem 1.

Let $(G,\mathbf{w})$ be a conservative graph. For any $x,y \in V(G)$, denote by $\lambda_{\mathbf{w}}(x,y)$ the minimum weight of a path connecting x and y. For $x \in V(G)$, let $m = m(x) = \min\{\lambda_{\mathbf{w}}(x,v) : v \in V(G)\}, M = M(x) = \max\{\lambda_{\mathbf{w}}(x,v) : v \in V(G)\}, V^i = V^i(x) = \{v \in V(G) : \lambda_{\mathbf{w}}(x,v) = i\}, G^i = G^i(x) = G[\cup_{j=m}^{i} V^j], \tilde{G}^i = G^i - E(G[V^i])$. Let $\mathcal{D}^i = \mathcal{D}^i(x)$ be the collection of vertex sets of components of G^i and $\mathcal{Q}^i = \mathcal{Q}^i(x)$ be the collection of vertex sets of components of $\tilde{G}^i$. Set $\mathcal{D} = \mathcal{D}(x) = \cup_{i=m}^{M} \mathcal{D}^i, \mathcal{Q} = \mathcal{Q}(x) = \cup_{i=m}^{M} \mathcal{Q}^i$. Let $\mathcal{R} = \mathcal{R}(x) = \mathcal{D} \cup \mathcal{Q}$, where the union is understood with multiplicity, i.e., if $R \in \mathcal{R}$ is an element of both $\mathcal{D}$ and $\mathcal{Q}$, then it is contained twice in the family $\mathcal{R}$.

The assertions of the second theorem are special cases of Theorem 4.4 and Lemma 5.7 in [4].

Theorem (Sebö). *Let $(G,\mathbf{w})$ be a conservative graph and $x \in V(G)$. Then*

(s1) *if $vu \in E^-(\mathbf{w}), \lambda_{\mathbf{w}}(x,v) = \lambda_{\mathbf{w}}(x,u) = i$, then v and u are in different elements of $\mathcal{Q}^i$ (i.e. in different components of $\tilde{G}^i$);*
(s2) $|\delta(R) \cap E^-(\mathbf{w})| = 1$ *if $x \notin R \in \mathcal{R}(x)$,*
$|\delta(\mathcal{R}) \cap E^-(\mathbf{w})| = 0$ *if $x \in R \in \mathcal{R}(x)$;*

Moreover, suppose that $m = m(x) < 0$. Let $D \in \mathcal{D}^m(x), r \in D$ be the endpoint of the edge in $\delta(D) \cap E^-(\mathbf{w})$ (which is unique by (s2)) and let $\mathbf{w}'$ be the weighting $\mathbf{w}|_{G[D]}$. Then

(s3) *for any $v \in D, \lambda_{\mathbf{w}'}(r,v) = 0$.* ∎

Let G be a graph and $\mathcal{P} = \{X_1,\ldots,X_k\}$ be a partition of $V(G)$. Denote by $G\langle\mathcal{P}\rangle$ the graph with the vertex set $\mathcal{P}$ and the edge set $E(G)$ which is obtained from G by shrinking every subset $X \in \mathcal{P}$ into a single vertex. For $X \subseteq V(G)$, denote by $\pi(X)$ the partition of $V(G)$ whose members are X and $|V(G) \setminus X|$ singletons.

We shall need the following easy consequence of Theorem 2.

Corollary 1. *Suppose that the assumptions of Theorem 2 hold. Then*

(c1) $E^-(\mathbf{w})$ *forms a perfect matching in $G[D]-r$;*
(c2) $G[D]$ *is factor-critical*;

(c3) $(G\langle \pi(N_G(D))\rangle, \mathbf{w})$ *is a conservative graph.*

Proof. Note that, for any $v \in D, \{v\} \in \mathcal{Q}^m$. Since $m < 0$ it follows (by (s2) of Theorem 2) that $|\delta(v) \cap E^-(\mathbf{w})| = 1$ for all $v \in D$. Combining this with the definition of r, we arrive at (c1).

Using (c1) and assertion (s3) of Theorem 2 we obtain that, for any $v \in D$, there exists an alternating (with respect to $E^-(\mathbf{w}) \cap E(G[D])$) path of even length connecting r with v in $G[D]$. It follows that $G[D]$ is factor-critical.

Now let $A = N_G(D)$ and $G' = G\langle \pi(A)\rangle$. Note that by the definition of $\mathcal{R}$ for each $R \in \mathcal{R}$, either $A \subseteq R$ or $A \cap R = \emptyset$. Therefore shrinking A into a vertex would not change the set of cuts $\mathcal{M} = \{\delta(R) : R \in \mathcal{R}\}$.

By (s2) of Theorem 2 any cut $M \in \mathcal{M}$ contains at most one edge of $E^-(\mathbf{w})$, by the definition of $\mathcal{R}$ each edge $e \in E(G')$ belongs to at most two members of $\mathcal{M}$, and by Theorem 2, each edge $e \in E^-(\mathbf{w})$ belongs to exactly two members of $\mathcal{M}$. We claim that this provides the conservativeness of $(G', \mathbf{w})$. Indeed, let C be the edge set of a cycle in G'. Next, let $M_1, \ldots, M_k$ be the members of $\mathcal{M}$ meeting C. Then by above, $\sum_{i=1}^{k} |C \cap M_i \cap E^-(\mathbf{w})| \leq k, \sum_{i=1}^{k} |C \cap M_i \cap E^-(\mathbf{w})| = 2|C \cap E^-(\mathbf{w})|$ and $\sum_{i=1}^{k} |C \cap M_i| \leq 2|C|$. Since $\sum_{i=1}^{k} |C \cap M_i| \geq 2k$, we obtain $|C| \geq 2|C \cap E^-(\mathbf{w})|$. ∎

3. THE MAIN RESULT

Let $(G, \mathbf{w})$ be a conservative graph. Denote by $\Pi(G, \mathbf{w})$ the set of partitions $\mathcal{P}$ of $V(G)$ such that $(G\langle \mathcal{P}\rangle, \mathbf{w})$ is conservative. Denote by $\Pi^*(G, \mathbf{w})$ the set of roughest partitions in $\Pi(G, \mathbf{w})$ (i.e., those which are not subpartitions of any member of $\Pi(G, \mathbf{w})$). Clearly, $\Pi(G, \mathbf{w}) \supseteq \Pi^*(G, \mathbf{w}) \neq \emptyset$ for any conservative graph $(G, \mathbf{w})$. We say that a partition $\mathcal{P}$ is *tree-like* if $G\langle \mathcal{P}\rangle$ has no cycles consisting of more than two edges. Note that if $\mathcal{P} \in \Pi^*(G, \mathbf{w})$ is tree-like then $E^-(\mathbf{w})$ induces a spanning tree of $G\langle \mathcal{P}\rangle$. Moreover, it is clear that if $\Pi(G, \mathbf{w})$ contains a tree-like partition then so does $\Pi^*(G, \mathbf{w})$.

In the following theorem Sebö (private communication) proved (b) ⇒ (c) and conjectured (c) ⇒ (a) and (c) ⇒ (b).

Theorem 3 . *Let G be a undirected connected graph. The following conditions are equivalent:*

- **(a)** *there exist a conservative weighting* $\mathbf{w}$ *and* $\mathbf{w}$*-zero cycles* C_1 *and* C_2 *such that the graph* $C_1 \cup C_2$ *is either an odd* K_4 *or an odd prism;*
- **(b)** *there exist a conservative weighting* $\mathbf{w}$ *and* $\mathbf{w}$*-zero cycles* C_1, C_2 *such that the graph* $C_1 \cup C_2$ *is non-bipartite;*
- **(c)** G *is not a Seymour graph;*
- **(d)** *there exists a conservative weighting* $\mathbf{w}$ *such that* $\Pi^*(G, \mathbf{w})$ *contains no tree-like partition.*

Proof. We shall prove (a) ⇒ (b) ⇒ (c) ⇒ (d) ⇒ (a).

(a) ⇒ (b). Obvious.

(b) ⇒ (c). Suppose to the contrary that G is a Seymour graph. Let C be an odd cycle of $H = C_1 \cup C_2$. Set $T = \{v \in V(G) : d_{E^-}(v) \text{ is odd}\}$, where $E^- = E^-(\mathbf{w})$. Let $\{D_1, \ldots, D_\nu\}$ be a collection of edge-disjoint T-cuts of G with $\nu = \nu(G, T)$. Since G is a Seymour graph and $E^-(\mathbf{w})$ is a minimum T-join by Guan's lemma, each D_i contains exactly one edge $e \in E^-(\mathbf{w})$. Therefore, D_i has in common with each $\mathbf{w}$-zero cycle either no edges or exactly two edges (in the latter case, one is in $E^-(\mathbf{w})$ and one is not). Thus, $E(C) \subseteq \cup_{i=1}^{\nu} D_i$. But $|E(C) \cap D_i|$ is even for all $i = 1, \ldots, \nu$. It follows that $|E(C)|$ is even, a contradiction.

(c) ⇒ (d). Let $T \subseteq V(G)$ be even and F be a minimum T-join. Then by Guan's lemma $\mathbf{w}_F$ is a conservative weighting of G. If (d) is false then $\Pi^*(G, \mathbf{w}_F)$ contains a tree-like partition $\mathcal{P}$. This means that $G\langle\mathcal{P}\rangle$ is a multi-tree (i.e., can be obtained from a tree by multiplying edges and adding loops). It remains to observe that the T-cuts in G corresponding to the (multi)edges of $G\langle\mathcal{P}\rangle$ form a packing of size $|E^-(\mathbf{w})|$. It follows that G is a Seymour graph, a contradiction.

(d) ⇒ (a). The proof relies on the following lemma which is to be proved in the remainder of the paper. ∎

Lemma 1. *Let G be a connected graph and $X \subseteq V(G)$ be a cut set of G. Suppose that $G - X$ has a factor-critical component with vertex set D such that $N_G(D) = X$. If $G\langle\pi(X)\rangle - D$ satisfies the condition (a) of Theorem 3 then so does G.*

Suppose that the implication does not hold and a graph G is a counterexample with the minimum number of vertices. That is

$$G \text{ satisfies (d) (with a weighting } \mathbf{w}) \text{ but does not satisfy (a)} \tag{2}$$

and

$$\text{(d)} \Rightarrow \text{(a)} \qquad \text{for each graph } H \text{ with} |V(H)| < |V(G)|. \tag{3}$$

Obviously,

$$|E^-(\mathbf{w})| \geq 2. \tag{4}$$

Claim. *Let Y be a nonempty subset of $V(G)$ such that $\pi(N_G(Y)) \in \Pi(G, \mathbf{w})$. If Y induces a factor-critical subgraph of G then $Y \cup N_G(Y) = V(G)$.*

Assume that $V(G) \setminus (Y \cup N_G(Y)) \neq \emptyset$ and Y has minimum cardinality over all sets satisfying this property.

Denote $H = G[Y]$, $A = N_G(Y)$ and $G' = G\langle\pi(A)\rangle - Y$. Let $\mathbf{w}'$ be the weighting $\mathbf{w}|_{E(G')}$.

Note that G' does not satisfy (d), for otherwise by (3) it satisfies (a) and, consequently, by Lemma 1, so does G itself. Hence $\Pi^*(G', \mathbf{w}')$ contains a tree-like partition $\mathcal{P} = \{X_1, \ldots, X_k\}$.

If $E^-(\mathbf{w}') = \emptyset$ then for any $v \in V(G) \setminus (A \cup Y)$, $(G\langle\pi(N_G(v))\rangle, \mathbf{w})$ is conservative (since already $(G\langle\pi(V(G) \setminus Y)\rangle, \mathbf{w})$ is conservative). By the minimality of $|Y|$, it follows that $|Y| = 1$ which contradicts (4).

Let $E^-(\mathbf{w}') \neq \emptyset$. In a pendant vertex X_i of $G'\langle\mathcal{P}\rangle$ not containing A, choose $v \in X_i$ incident to the edge $e \in E^-(\mathbf{w}') \cap \delta(X_i)$. Then $(G\langle\pi(N_G(v))\rangle, \mathbf{w})$ is conservative, and we have $|Y| = 1$ again. Hence $\{X_1, \ldots, X_k, Y\}$ is in $\Pi(G, \mathbf{w})$ and tree-like, a contradiction.

Let $xy \in E^-(\mathbf{w})$. Set $m = m(x)$ and consider $D \in \mathcal{D}^m$. Since $m(x) \leq -1, x \notin D$. Let r be the endpoint in D of the edge in $\delta(D) \cap E^-(\mathbf{w})$. By Corollary 1, (c1′) $V(G[D]) \cap E^-(\mathbf{w})$ is a matching in $G[D]$; (c2′) $G[D]$ is factor-critical; and (c3′) $\pi(N_G(D)) \in \Pi(G < \mathbf{w})$. By (c2′), (c3′) and the claim, $D \cup N_G(D) = V(G)$. Assertion (c3′) implies that $\mathbf{w}(e) = \pm 1$ for each edge $e \in E(G)$ with both ends in $N_G(D)$. Since $x \notin D$, it follows that $y \in D$ and, in addition, by (s2) of Theorem 2 the only edge in $E^-(\mathbf{w})$ with exactly one end in D is xy. Consequently, $r = y$ and $m = -1$. This together with (c1′) implies that $E^-(\mathbf{w})$ is a matching and $T(\mathbf{w}) = D \cup \{x\}$. Since by (c2′), $G[T(\mathbf{w})] - x$ is factor-critical and x was chosen arbitrarily from $T(\mathbf{w})$, we conclude that

$G[T(\mathbf{w})]$ is bicritical and thereby 1-extendable. It is an easy observation that any bicritical graph with more than two vertices is non-bipartite. Therefore, by (4), $G[T(\mathbf{w})]$ is non-bipartite. Now applying Theorem 1 we obtain that G contains an even subdivision H of either K_4 or triangular prism. In either cases the edges of H can be partitioned into three matchings M_1, M_2 and M_3 so that $M_1 \cup M_2$ and $M_1 \cup M_3$ are Hamiltonian cycles in H. Thus G satisfies (a) with $\mathbf{w} = \mathbf{w}_{M_1}$, a contradiction. ∎

4. PRELIMINARY OBSERVATIONS

In this section we state several easy observations to be referred to in the proof of Lemma 1.

Proposition 1. *Let G be a graph and $F \subseteq E(G)$. If F is a matching in G then $\mathbf{w}_F$ is a conservative weighting.*

Proposition 2. *Let G be a graph and $\mathbf{w}$ be a weighting of G. The weighting $\mathbf{w}$ is conservative if and only if the weighting $\mathbf{w}|_{E(B)}$ is conservative for every block B* of *G.*

Proposition 3. *Let G be a graph. If $\mathbf{w}$ is a conservative weighting of $G\langle\pi(X)\rangle$ for some $X \subseteq V(G)$ then $\mathbf{w}$ is a conservative weighting of G.*

Denote by $\mathcal{O}_1$ and $\mathcal{O}_2$ the sets of odd K_4-s and odd prisms respectively. Let $\mathcal{O}_1^e(\mathcal{O}_2^e)$ denote the set of even subdivisions of K_4 (triangular prism). Clearly, $\mathcal{O}_k^e \subset \mathcal{O}_k$ $k = 1, 2$. Let $\mathcal{O} = \mathcal{O}_1 \cup \mathcal{O}_2$ and $\mathcal{O}^e = \mathcal{O}_1^e \cup \mathcal{O}_2^e$.

Proposition 4. *If a graph G has a subgraph $H \in \mathcal{O}^e$ then G satisfies the condition (a) of Theorem 3.*

Proposition 5. *Let $G \in \mathcal{O}^e$ and $f \in E(G)$. Then there exists a perfect matching M in G such that $f \in M$ and G is the union of two $\mathbf{w}_M$-zero cycles. Moreover, if f is incident with a vertex of degree 3, then M and $\mathbf{w}_M$-zero cycles can be chosen in such a way that f lies in both.*

Proposition 6. *Let $G \in \mathcal{O}$ and let e_1, e_2 be adjacent edges of G. Then G has an even cycle passing through e_1 and e_2.*

Proposition 7. *Let $G \in \mathcal{O}_2$ and let $u \in V(G)$ be a vertex of degree 3. Then G has three disjoint paths of the same parity connecting u with some vertex $v \in V(G)$. If, in addition, $G \in \mathcal{O}_2^e$ then G has three disjoint paths of odd length connecting u with some vertex $v \in V(G)$.*

5. PROOF OF LEMMA 1

Let $L = V(G)\backslash(X \cup D)$. Denote by x^* the contracted vertex of $G\langle\pi(X)\rangle$. Let $\tilde{G} = G\langle\pi(X)\rangle - D$ and $G^* = G\langle\pi(X)\rangle - L$. Let $\mathbf{w}$ be a conservative weighting of G and let C_1, C_2 be $\mathbf{w}$-zero cycles such that $\tilde{H} = \tilde{C}_1 \cup \tilde{C}_2 \in \mathcal{O}$. Denote by H (respectively, $C_k, k = 1, 2$) the subgraph of G spanned by the edges of $\tilde{H}$ (respectively, $\tilde{C}_k$). Let $X \cap V(H) = \{v_1, \ldots, v_l\}$. As $d_{\tilde{H}}(x^*) \leq 3, l \leq 3$. If $l \leq 1$ then $H = \tilde{H}$, so we may assume that $l \in \{2, 3\}$. Since $N_G(D) = X$, for every $k \in \{1, \ldots, l\}$, there is an edge $f_k \in E(G)$ connecting v_k with D. Since $G[D]$ is factor-critical, G^* has perfect matchings $F_k \subset E(G^*)$ such that $f_k \in F_k, k = 1, \ldots, l$. Let $\tilde{S}$ be the component of the subgraph of G^* spanned by $\cup_{k=1}^{l} F_k$ which contains x^*. Let S denote the subgraph of G spanned by $E(\tilde{S})$. It is obtained from $\tilde{S}$ by splitting x^* into $v_1, \cdots v_l$. The components of the graph spanned by the union $F_1 \cup F_j, 2 \le j \le l$, are cycles and single edges and whence that containing x^* is a cycle. It follows that S is a connected subgraph of G.

Case 1. $l = 2$.
Then $\tilde{S}$ is an even cycle and S is an even path whose ends are v_1 and v_2. Let $M = F_1 \cap E(S)$. Set

$$\mathbf{w}^*(e) := \begin{cases} \mathbf{w}(e) & \text{if } e \in E(\tilde{G}), \\ -1 & \text{if } e \in M, \\ +1 & \text{otherwise.} \end{cases} \tag{5}$$

By Propositions 1 and 2, $\mathbf{w}^*$ is a conservative weighting of $G\langle\pi(X)\rangle$ and, consequently, by Proposition 3 it is that of G. Since S is an even path, $H \cup S$ belongs to $\mathcal{O}$. Note that $H \cup S = C_1 \cup C_2 \cup S$. We may have that either C_1, C_2 are both paths or exactly one of them, say C_1, is a path while C_2 is a cycle. In the former case $C_1 \cup S$ and $C_2 \cup S$ are the desired $\mathbf{w}^*$-zero cycles, otherwise $C_1 \cup S$ and C_2 are those.

From now on we assume that $l = 3$. It follows that exactly one vertex v_i, say v_3, is incident with an edge which is contained in both cycles $\tilde{C}_1$ and $\tilde{C}_2$. In other words, $X \cap V(C_k) = \{v_k, v_3\}$, $k = 1, 2$.

Case 2. $l = 3$ *and* $\tilde{S}$ *is bipartite*.
We show first that S has three disjoint paths $P_k, k = 1, 2, 3$ of odd length connecting v_k with some $v \ne v_k$, $k = 1, 2, 3$. Indeed, let R_k, $k = 1, 2$, denote the path consisting of edges in $F_k \cup F_3$ and connecting v_k with v_3. Choose the first vertex v on R_2 which lies on R_1. Define P_k, $k = 1, 3$, to be the subpaths of R_1 connecting v with v_k, and P_2 to be the subpath of R_2 connecting v with v_2. By construction, P_k are pairwise disjoint and, since $\tilde{S}$ is bipartite, they have odd length. Now let $Q = P_1 \cup P_2 \cup P_3$. The matching $M = F_3 \cap E(Q)$ covers all the vertices of Q except v_1 and v_2. Define $\mathbf{w}^*$ by the equation (5). Again, by Propositions 1 and 2, $\mathbf{w}^*$ is a conservative weighting of $G\langle\pi(X)\rangle$, and whence by Proposition 3, it is that of G. Furthermore, $H \cup Q$ belongs to $\mathcal{O}$ being the union of $\mathbf{w}^*$-zero cycles $C_1 \cup P_1 \cup P_3$ and $C_2 \cup P_2 \cup P_3$.

Case 3. $l = 3$ *and* $\tilde{S}$ *is non-bipartite.*
Note that $\tilde{S}$ is 1-extendable. By Theorem 1, it follows that $\tilde{S}$ has a subgraph $\tilde{Q} \in \mathcal{O}^e$. Let Q be the subgraph of G spanned by $E(\tilde{Q})$. Note that $V(Q) \cap X \subseteq \{v_1, v_2, v_3\}$. If $|V(Q) \cap X| \le 1$ then $Q = \tilde{Q}$, and the conclusion follows by Proposition 4. If $|V(Q) \cap X| = 2$, we obtain the desired conclusion using Propositions 4, 6 and the argument of Case 1. Thus we may assume further that $V(Q) \cap X = \{v_1, v_2, v_3\}$. By Proposition 5 there exists a perfect matching M of $\tilde{Q}$ and $\mathbf{w}_M$-zero cycles $\tilde{D}_1$ and $\tilde{D}_2$ such that $f_3 \in M$, $f_3 \in E(\tilde{D}_1) \cap E(\tilde{D}_2)$ and $\tilde{Q} = \tilde{D}_1 \cup \tilde{D}_2$. Let D_k, $k = 1, 2$, be the subgraph of G spanned by $E(\tilde{D}_k)$. Note that $V(D_k) \cap X = \{v_k, v_3\}$, $k = 1, 2$.

Subcase 3.1. $\tilde{Q} \in \mathcal{O}_2^e$.
By Proposition 7, Q contains three disjoint paths P_k, $k = 1, 2, 3$, having odd length and such that P_k connects v_k with some $v \in V(Q)$, $v \ne v_k$, $k = 1, 2, 3$. It remains to apply the argument of Case 2 arriving at the same conclusion.

Subcase 3.2. $\tilde{Q} \in \mathcal{O}_1^e$, $\tilde{H} \in \mathcal{O}_2$.
By Proposition 7, H contains three disjoint paths P_k, $k = 1, 2, 3$, of the same parity and such that P_k connects v_k with some $v \in V(H)$, $v \ne v_k$, $k = 1, 2, 3$. If each P_k is odd, the desired conclusion is obtained by repeating the argument of Case 2. If all these paths are of even length, then $S \cup P_1 \cup P_2 \cup P_3 \in \mathcal{O}_1^e$ and the desired conclusion follows from Proposition 4.

Subcase 3.3. $\tilde{Q} \in \mathcal{O}_1^e$, $\tilde{H} \in \mathcal{O}_1$.

Note first that $H \cup Q$ is an odd prism. Define $\mathbf{w}^*$ by the equation (5). By Propositions 1 and 2, $\mathbf{w}^*$ is a conservative weighting of $G\langle \pi(X) \rangle$ and thereby by Proposition 3 it is that of G. Finally, $H \cup Q$ is the union of $\mathbf{w}^*$-zero cycles $D_1 \cup C_1$ and $D_2 \cup C_2$, as desired. ∎

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