

Total Colorings of Planar Graphs with Large Maximum Degree

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Abstract: It is proved that a planar graph with maximum degree $\Delta \geq 11$ has total (vertex-edge) chromatic number $\Delta + 1$. © 1997 John Wiley & Sons, Inc. J Graph Theory 26: 53–59, 1997

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1. INTRODUCTION

The *total chromatic number* $\chi'' = \chi''(G)$ of a graph G is the smallest number of colors that suffice to color the vertices and edges of G so that no two adjacent or incident elements have the same color. It is clear that $\chi'' \geq \Delta + 1$, and Behzad [1] and Vizing [12] conjectured that

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$\chi'' \leq \Delta + 2$, for every graph G with maximum degree Δ . This conjecture was verified by Rosenfeld [10] and Vijayaditya [11] for $\Delta = 3$ and by Kostochka [7, 8, 9] for $\Delta \leq 5$. For planar graphs the conjecture was verified by Borodin [2] for $\Delta \geq 9$, and now remains open only for $6 \leq \Delta \leq 7$ [6]. But for planar graphs it is often possible to obtain the exact result $\chi'' = \Delta + 1$. We proved this in [3] for planar graphs with various combinations of maximum degree and girth. With no condition on the girth, Borodin [2] proved it for planar graphs with maximum degree $\Delta \geq 14$. In [4] we improved this to $\Delta \geq 12$ (and, moreover, with the list total chromatic number χ''_{list} in place of χ''). This paper is devoted to a proof of the following further improvement for χ'' .

Theorem. If G is a planar graph with maximum degree $\Delta \geq 11$, then $\chi''(G) = \Delta + 1$.

In fact, we prove that if $\Delta \geq 11$ then $\chi''(G) \leq \Delta + 1$ for all planar graphs with maximum degree *at most* Δ , and we let $G = (V, E)$ be a counterexample to this more general result with $|V| + |E|$ as small as possible, so that every proper subgraph of G is totally $(\Delta + 1)$ -colorable. It is easy to see that G is 2-connected, and hence has no vertices of degree 1.

2. REDUCIBLE CONFIGURATIONS

In this section we start the proof of the theorem by obtaining structural information about our minimal counterexample G , which shows that certain configurations are *reducible*, that is, they cannot occur in G . The degree of vertex v is denoted by $d(v)$, and a d -vertex is a vertex v with $d(v) = d$.

Lemma 1. (a) If uw is an edge of G with $d(u) \leq 5$, then $d(u) + d(w) \geq \Delta + 2 \geq 13$ (and so 2-vertices of G are adjacent only to Δ -vertices).

(b) The graph of all edges joining 2-vertices to Δ -vertices is a forest.

Proof. If G contains an edge uw with $d(u) \leq 5$ and $d(u) + d(w) \leq \Delta + 1$, then we can totally color $G - uw$ with $\Delta + 1$ colors (by the minimality of G), erase the color on u , and then color uw and u in turn, since the number of colors that we may not use is at most $(\Delta - 1) + 1 = \Delta$ for uw and $5 + 5 = 10 < \Delta$ for u . This contradicts the choice of G as a counterexample and so proves (a).

If (b) is false then G contains a cycle $v_1v_2 \cdots v_{2k}v_1$ of even length such that $d(v_1) = d(v_3) = \cdots = d(v_{2k-1}) = 2$. Then, by the minimality of G , we can totally color $G - \{v_1, v_3, \dots, v_{2k-1}\}$ with $\Delta + 1$ colors. Each edge of C now has at most $(\Delta - 2) + 1 = \Delta - 1$ colors that may not be used on it, hence at least two that may, and so the problem of coloring the edges of C is equivalent to coloring the vertices of an even cycle, given a choice of two colors at each vertex; it is well known [5, 13] that this is possible. The 2-vertices of C are now easily colored, and this contradicts the choice of G as a counterexample to the theorem, thereby proving (b). ■

In Figure 1, vertices marked \bullet have no edges of G incident with them other than those shown, and so have degree 2 or 3 in G .

Lemma 2. G has no subgraph isomorphic to either of the configurations in Figure 1 (a) and (b).

Proof. Suppose that G does contain one of these configurations. Choose a total coloring of $G - uv$ with $\Delta + 1$ colors, which exists by the minimality of G , and erase the colors on the vertices marked \bullet . If we can prove that the resulting partial total coloring c can be extended to edge uv , then we will have a contradiction to the choice of G as a counterexample to the theorem,

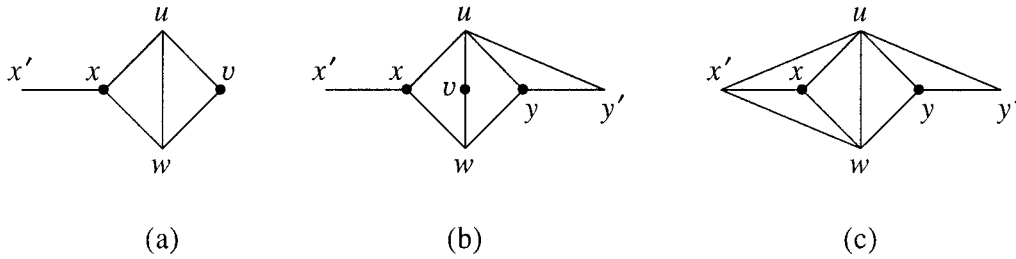


FIGURE 1. Some reducible configurations.

since vertices with degree at most 3 are easily colored. Let $c(vw) = 1$. We may suppose that color 1 is not present at u (that is, $1 \neq c(u)$ or $c(e)$ for any edge e incident with u), since if it is then some other color is not present at u and can be given to uw . The proof now divides.

(a) Since $1 = c(vw)$ is not present at u , the three edges in the path $vwux$ have three different colors, say 1, 2, 3 (in that order). Recolor these edges 1, 2, 1 or 2, 1, 2 according as $c(xx') = 2$ or $\neq 2$, and color uw with 3.

(b) We may suppose that $c(xx') = c(yy') = 1$, since otherwise we can give color $c(ux)$ or $c(uy)$ to uw and recolor ux or uy with 1. Thus the edges of path $vwyy'u$ have colors (w.l.o.g.) 1, 2, 1, 3 or 1, 2, 1, 2. In the former case, recolor these edges 1, 2, 3, 1 and color uw with 3. In the latter case, recolor the edges of path $vwyy'ux$ with 2, 1, 2, 1, 2 and give color $c(ux)$ to uw . ■

Lemma 3. G has no subgraph isomorphic to the configuration in Figure 1(c).

Proof. Suppose it has. Let C denote the 4-cycle $uxwyu$. Color $G - E(C)$ and erase the colors on x, y . Choose this coloring c if possible so that $c(xx') \neq c(yy')$ and $c(uw) \in \{c(xx'), c(yy')\}$. We now show that this is possible.

Suppose first that $c(xx') = c(yy') = 1$, say. If color 1 is present at both u and w then there are at least two colors available for each edge of C , which leads to a contradiction since an even cycle is list-edge-2-colorable (and x and y are then easily colored). Thus color 1 is absent at u or w , and we can ensure $c(xx') \neq c(yy')$ by swapping the color of xx' with that of $x'u$ or $x'w$ respectively. So from now on, let $c(xx') = 1, c(yy') = 2$.

Now suppose that $c(uw) \notin \{1, 2\}$. If we cannot change $c(uw)$ to 1 or 2 then each of these colors is present at at least one of u, w . If both are present at u (say), then two other colors are absent at u and we can color wx, wy first and then ux, uy . If however 1 (only) is present at u and 2 (only) at w , then after coloring ux, wy with 2, 1, there remains a color for each of uy, wx . This contradiction shows that we may suppose $c(uw) \in \{1, 2\}$.

Let $O(v)$ denote the set of colors absent at vertex v . If $c(uw) = 1$, it is now easy to complete the coloring unless $|O(u)| = |O(w)| = 2$ and $2 \in O(u) \cap O(w)$, in which case we might have to give color 2 to both ux and wx . The same applies with 1 and 2 interchanged. In either case, $|O(u) \setminus \{1, 2\}| = |O(w) \setminus \{1, 2\}| = 1$, say $O(v) \setminus \{1, 2\} = \{\bar{c}(v)\}$ ($v = u, w$). By symmetry (since we shall make no further use of $x'w$) we may suppose that $c(uy') = 3 \neq \bar{c}(w)$. Now we can color or recolor the edges of the trail $wyy'uxwuy$ with colors 2, 3, 2, 3, $\bar{c}(w), 1, \bar{c}(u)$. Finally, x and y are easily colored. ■

Lemma 4. No face of G has more than one 2-vertex in its boundary.

Proof. Suppose that v, x are 2-vertices in the boundary of a face f . If v, x are separated by at least two other vertices each way round the boundary, then we can simply identify them; the resulting simple planar graph is totally $(\Delta + 1)$ -colorable by the minimality of G , and this gives a

total- $(\Delta + 1)$ -coloring of G . So suppose that u, v, w, x, y are consecutive vertices in the boundary of f . By Lemma 1 (a) and (b), $u \neq y$. So identifying v with x and deleting one of the edges vw and wx , say vw , gives a simple planar graph, which has a total- $(\Delta + 1)$ -coloring c by the minimality of G . Apply this coloring to G . If $c(xy)$ is absent at w , give this color to edge vw ; if not, give edge vw color $c(wx)$ and recolor wx with a color that is absent at w (and so different from $c(xy)$), finally recoloring x if necessary. ■

3. DISCHARGING

We shall complete the proof of the theorem by using discharging in order to obtain a contradiction. Let (V, E, F) be a plane embedding of G . We assign a ‘‘charge’’ $M(\phi)$ to each element $\phi \in V \cup F$, where

$$M(\phi) := \begin{cases} d(\phi) - 6 & \text{if } \phi \in V, \\ 2r(\phi) - 6 & \text{if } \phi \in F, \end{cases}$$

where $r(f)$ denotes the number of edges around face f . Euler's formula $|V| - |E| + |F| = 2$ can be rewritten in the form $(2|E| - 6|V|) + (4|E| - 6|F|) = -12$, which implies that

$$\sum_{\phi \in V \cup F} M(\phi) = \sum_{v \in V} (d(v) - 6) + \sum_{f \in F} (2r(f) - 6) = -12. \tag{1}$$

We shall now redistribute the charge, without changing its sum, in such a way that the sum is provably nonnegative, and this contradiction will prove the theorem. Before stating the redistribution rules, we need some definitions.

A 4-face $uvxy$ is *special* if there is a 3-face vxy (and hence $d(x) = 2$). We shall suppose that the embedding of G is chosen so as to minimize the number of special 4-faces. This implies that

$$\text{for each special 4-face } uvxy, \text{ the edge } vy \text{ separates two triangles,} \tag{2}$$

since otherwise we could reduce the number of special 4-faces by moving edge vy across the path vxy .

A *lo-face* of G is a 3-face or a special 4-face; a *hi-face* is any other face, that is, a nonspecial 4-face or a ≥ 5 -face. A *lo-triangle* of G is a 3-face in the graph obtained by deleting all 2-vertices of G and their incident edges; thus by Lemma 4 it is either a 3-face of G having no vertices of degree 2, or a triangle uvw containing a 3-face vxy and a special 4-face $uvxy$ with $d(x) = 2$.

By Lemma 1 (a) and (b), it is possible to find a matching in G that pairs off all the 2-vertices with some of the Δ -vertices: in each component of the forest in Lemma 1(b), choose a Δ -vertex v , and match each 2-vertex u with the Δ -vertex w adjacent to u that is further from v . (Note that the endvertices of this forest are all Δ -vertices.) We call w the *master* of u and u the (unique) *2-dependant* of w .

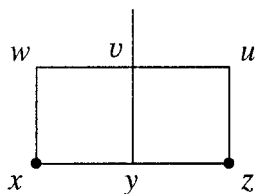


FIGURE 2. Vertex v is a sponsor of vertex y .

If the configuration in Figure 2 occurs in G , where $d(x) = d(z) = 2$, $d(v) \geq 4$ and neither of the 4-faces $vwxy$ and $vuzy$ is special, then we say that v is a *sponsor* of y . Note that y in turn can sponsor other vertices, and even cycles of sponsorship are possible.

The rules for redistribution of charge are as follows.

- R1:** A special 4-face gives charge 2 to its 2-vertex. A nonspecial 4-face gives 1 to an incident ≤ 5 -vertex and 1 to the vertex opposite it. A ≥ 5 -face gives 3 to a 2-vertex and 1 to any other ≤ 5 -vertex.
- R2:** Suppose $d(v) = 2$. If v is incident with a ≥ 5 -face, then v receives 1 from its master. Otherwise, v receives $13/10$ from its master and $7/10$ from its other neighbor.
- R3:** Suppose $d(v) = 3$. If v is not incident with hi-faces, then v receives 1 from each of its neighbors. If v is incident with precisely one hi-face $\cdots avb \cdots$ then v receives $3/5$ from each of a, b and $4/5$ from its third neighbor. If v is incident with precisely two hi-faces $\cdots avb \cdots$ and $\cdots bvc \cdots$ then v receives $1/2$ from each of a, c .
- R4:** If $d(v) = 4$ or 5 then v receives $1/2$ along every incident edge that separates two lo-faces.
- R5:** A vertex gives $1/5$ to any vertex of which it is a sponsor.
- R6:** If $f = vxwy$ is a nonspecial 4-face with $d(x) = 2$ and $d(y) \geq 10$, so that f gives 1 to y by R1, then y gives $1/2$ to each of v and w .

Let $M^*(\phi)$ be the resulting charge on ϕ , so that $\sum_{\phi \in V \cup F} M^*(\phi) = -12$ by (1). We shall obtain a contradiction by proving that $M^*(\phi) \geq 0$ for each element ϕ . We start with the easy cases.

Suppose first that ϕ is a face f . If $r(f) = 3$ then $M^*(f) = M(f) = 0$. If $r(f) = 4$ then $M(f) = 2$ and (by Lemma 1(a)) f gives up at most 2 by R1. If $r(f) = r \geq 5$ then $M(f) = 2r - 6$ and (by Lemma 4) f gives up at most $2 + \lfloor \frac{1}{2}r \rfloor$ by R1. Thus $M^*(f) \geq 0$ in all cases.

Suppose now that ϕ is a vertex v . If $d(v) = 2$ then $M(v) = -4$. If v is incident with a ≥ 5 -face f then v receives 3 from f by R1 and 1 from its master by R2. The only other possibilities (since G is simple) are that v is between two (nonspecial) 4-faces, or between a (special) 4-face and a 3-face, in which case v receives 2 from the 4-face(s) by R1 and 2 from neighboring vertices by R2. In either case, $M^*(v) = 0$.

If $d(v) = 3$ then $M(v) = -3$. By R1, v receives 1 from any incident hi-face, and it is easy to see from R3 that $M^*(v) = 0$.

If $d(v) = 4$ or 5 then $M(v) = -2$ or -1 . By R1, v receives 1 from any incident hi-face. By this and R4, it is easy to see that $M^*(v) \geq 0$ if v is not a sponsor. If v is a sponsor, then it receives 2 from the two hi-faces (non-special 4-faces) involved in the sponsorship. If there is another hi-face at v then v receives another 1 which suffices for five sponsorships; otherwise, the other faces are all lo-faces and v receives at least $\frac{1}{2}$ between them, which covers its one sponsorship.

If $d(v) = 6$ or 7 then $M(v) \geq 0$ and v gives nothing if it is not a sponsor. If v is a sponsor then it receives 2 from the two hi-faces (non-special 4-faces) involved in the sponsorship, and this suffices for $10 > 7$ sponsorships.

If $d(v) = 8$ or 9 then $M(v) \geq 2$ and v can give $\frac{1}{2}$ to at most four vertices by R4. If v is a sponsor then it receives an additional 2 from two hi-faces, which suffices for $10 > 9$ sponsorships.

If $d(v) \geq 10$ then, in addition to giving charge by R4 and R5, v can give charge to 3-vertices by R3 and (if $d(v) \geq 11$) to 2-vertices by R2. We shall deal with this case in the next section.

4. VERTICES WITH DEGREE AT LEAST 10

Let v be a fixed vertex with $d(v) \geq 10$. For the sole purpose of proving $M^*(v) \geq 0$, we carry out a temporary local redistribution of charge whereby some neighbors of v that have received

charge from v will give some of it to other neighbors of v that have received no charge from v . We shall prove that, after this local redistribution,

$$v \text{ gives } 1 \text{ to its } 2\text{-dependant (if any) and at most } \frac{2}{5} \text{ to each other neighbor.} \quad (3)$$

This will suffice to prove the result, since it shows that $M^*(v) \geq (d(v) - 6) - 1 - \frac{2}{5}(d(v) - 1) = \frac{3}{5}(d(v) - 11) \geq 0$ if $d(v) \geq 11$, while if $d(v) = 10$ then v is not adjacent to any 2-vertices by Lemma 1(a) and so $M^*(v) \geq (d(v) - 6) - \frac{2}{5}d(v) = 0$.

Let x be a neighbor of v that has received “too much” charge from v (that is, more than is allowed for in (3)). The following rules for the temporary local redistribution of charge cover all possibilities for x . (Note that v cannot be a sponsor of x if $d(x) < \Delta$, by Lemma 1(a) and the structure of Figure 2.)

L1: Suppose that x is a 2-vertex that has received $\frac{13}{10}$ or $\frac{7}{10}$ from v by R2. If edge vx separates a 3-face vxy from a special 4-face $uvxy$, then x now gives $\frac{3}{10}$ to y (and so retains 1 or $\frac{2}{5}$). The only other possibility is that vx separates two nonspecial 4-faces $wvxy$ and $wvxy$, in which case x now gives $\frac{3}{20}$ to each of u, w unless one has degree 3 and the other has larger degree, in which case x gives $\frac{3}{10}$ to whichever has larger degree. (The reason for this is that a 3-vertex could have received charge from v by R3. However, this cannot happen if both u and w have degree 3, since then neither belongs to a triangle by Lemma 2(b).)

L2: Suppose $d(x) = 3$. Suppose first that vx separates two lo-triangles vxy_1 and vxy_2 . If xy_1y_2 is a lo-triangle then x has received 1 from v by R3 and now gives $\frac{3}{10}$ to each of y_1, y_2 ; otherwise x has received $\frac{4}{5}$ from v by R3 and now gives $\frac{1}{5}$ to each of y_1, y_2 . Suppose now that vx separates a lo-triangle vxy from a hi-face. If there is another lo-triangle at x then x has received $\frac{3}{5}$ from v by R3 and now gives $\frac{1}{5}$ to y ; otherwise x has received $\frac{1}{2}$ from v by R3 and now gives $\frac{1}{10}$ to y . (If vx separates two hi-faces then x has received nothing from v .)

L3: Suppose that $d(x) = 4$ or 5 and x has received $\frac{1}{2}$ from v by R4. Then vx separates two lo-triangles vxy_1 and vxy_2 , and x now gives $\frac{1}{20}$ to each of y_1, y_2 .

Note that the redistribution in L2 and L3 always happens along an edge xy of a triangle vxy .

It is clear from L1–L3 that (3) holds for any neighbor of v that received charge from v by R1–R6, since no such neighbor receives anything more by L1–L3. So let y be a neighbor of v that receives charge by L1–L3. We must prove that it receives at most $\frac{2}{5}$.

Suppose first that y receives $\frac{3}{10}$ by L1 from a 2-vertex x adjacent to both v and y . Then the path vxy separates a 3-face vxy from a special 4-face $uvxy$, and by (2) the edge vy separates two 3-faces, say vxy and vwv . By Lemma 2(a), $d(u) \geq 4$ and $d(w) \geq 4$. By L3, y receives at most $\frac{1}{20}$ from each of u, w , and nothing from any other vertex except x , making a total of at most $\frac{3}{10} + \frac{2}{20} = \frac{2}{5}$.

Suppose now that y receives $\frac{3}{10}$ or $\frac{3}{20}$ from a 2-vertex x across a 4-face by L1, so that vx lies between nonspecial 4-faces $vxyw$ and $vxyz$, say. If $d(y) = 3$ then y receives $\frac{3}{20}$ from x and at most $\frac{3}{20}$ from a 2-vertex on the other side of vy , making less than $\frac{2}{5}$ in total. If $4 \leq d(y) \leq 9$ then y is not adjacent to 3-vertices by Lemma 1(a); it receives at most $\frac{3}{10}$ from x and nothing more unless it receives $\frac{3}{10}$ from a 2-vertex on the other side of vy , in which case y is a sponsor of v and gives back $\frac{1}{5}$ to v by R5, thus retaining exactly $\frac{2}{5}$ from v . If $d(y) \geq 10$ then y can receive up to $\frac{3}{10}$ from x and $\frac{3}{10}$ from a 2-vertex or 3-vertex on the other side of vy , but y gives $\frac{1}{2}$ to v by R6, and so retains at most $\frac{1}{10}$ from v .

Finally, suppose y receives nothing from 2-vertices. If y is sponsored by v then it receives $\frac{1}{5}$ and nothing more from v . Assume y is not sponsored by v . By L2 and L3, y can get

only $\frac{3}{10}$, $\frac{1}{5}$, $\frac{1}{10}$ or $\frac{1}{20}$ from at most two ≤ 5 -vertices x_1, x_2 lying in triangles vyx_1 and vyx_2 . To receive more than $\frac{2}{5}$ in total, y must receive $\frac{3}{10}$ from x_1 or x_2 , say x_1 , and either $\frac{3}{10}$ or $\frac{1}{5}$ from x_2 . By L2, there are three lo-triangles at x_1 and two or three lo-triangles at x_2 . But this contradicts Lemma 3, and this contradiction completes the proof of the Theorem.

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