Total Interval Number for Graphs with Bounded Degree

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Abstract: The total interval number of an n-vertex graph with maximum degree Δ is at most $(\Delta+1/\Delta)n/2$, with equality if and only if every component of the graph is $K_{\Delta,\Delta}$. If the graph is also required to be connected, then the maximum is $\Delta n/2+1$ when Δ is even, but when Δ is odd it exceeds $[\Delta+1/(2.5\Delta+7.7)]n/2$ for infinitely many n. © 1997 John Wiley & Sons, Inc. J Graph Theory **25:** 79–84, 1997

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Given sets $\{S_v : v \in V\}$, the *intersection graph* of the collection of sets is the simple graph with vertex set V such that u is adjacent to v if and only if $S_u \cap S_v \neq \emptyset$. The family of sets is an *intersection representation* of its intersection graph. The *interval graphs* are the intersection graphs representable by assigning each vertex a single interval on the real line. More generally,

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we allow a representation f to assign each vertex a union of intervals on the real line; if G is the intersection graph of this collection, then f is a *multiple-interval representation* of G. Let $\#f(v)$ be the number of disjoint intervals whose union is $f(v)$. If $\#f(v) = k$, we say that $f(v)$ *consists* of k intervals or that v is *assigned* k intervals.

We may try to make the representation of G ''efficient'' by minimizing $\max_{v \in V} \#f(v)$ or $\sum_{v \in V} \#f(v)$. The *interval number* $i(G)$ of a graph G is the minimum of $\max_{v \in V(G)} \#f(v)$ over all multiple-interval representations of G. Interval number has been studied since 1979, beginning with [7] and [2]. The *total interval number* $I(G)$ of a graph G is the minimum of $\sum_{v \in V(G)} \#f(v)$ over all multiple-interval representations of G. Although introduced in [2], total interval number was not studied until Aigner and Andreae [1] obtained extremal results for some fundamental families. Further results appear in [3, 4, 5, 6].

In this paper we prove that $I(G) \le (\Delta + 1/\Delta)n/2$ for graphs with maximum degree at most Δ, which is best possible. The proof yields a polynomial algorithm for producing a representation that satisfies the bound. Kratzke and West [5] proved that if G contains a collection of t pairwise edge-disjoint trails that together contain an endpoint of every edge of G, then $I(G) \le e(G) + t$, where $e(G) = |E(G)|$. Such a collection of trails is a *trail cover* of size t, generalizing the notion of vertex cover; in this paper, ''covering e''means ''containing an endpoint of e''. Let $t(G)$ denote the minimum size of a trail cover. If G is triangle-free, then $I(G) = e(G) + t(G)$; a simple counting argument [5] establishes the lower bound. The graph $mK_{\Delta,\Delta}$ has m components that are complete bipartite graphs; it is regular and triangle-free, with $n = 2m\Delta$ vertices, and its minimum trail covers have size m. Hence $I(mK_{\Delta,\Delta}) = m\Delta^2 + m = (\Delta + 1/\Delta)n/2$, and our bound is best possible. We also prove that these are the only graphs achieving the bound. The proof yields a polynomial-time algorithm to achieve the bound.

CONNECTED GRAPHS

Before proving the bound for general graphs, we discuss the more restricted class of connected graphs. For Δ even, the maximum of $I(G)$ in terms of n and Δ is $\Delta n/2 + 1$. For Δ odd, we provide constructions where the excess over $\Delta n/2$ is linear in n.

Proposition. Suppose Δ is even. Among connected n-vertex graphs with maximum degree Δ, the maximum of the total interval number is $\Delta n/2 + 1$.

Proof. Suppose G is connected and has maximum degree Δ. If G is Eulerian, then $I(G) \le e(G) + 1 \le \Delta n/2 + 1$, with equality if G is triangle-free and regular. If G is not Eulerian, suppose G has $2k$ vertices of odd degree. Since G is connected, we can decompose $E(G)$ into k trails, so $t(G) \le k$. Since each vertex of odd degree has degree less than Δ, we have $2e(G) \le \Delta n - 2k$, and hence $I(G) \le e(G) + k \le \Delta n/2$. ∎

In addition to the Δ-regular triangle-free graphs, equality holds also for Δ-regular graphs in which every vertex belonging to a triangle is a cut-vertex. When Δ is odd, the bound of $\Delta n/2 + 1$ no longer holds; we provide a construction.

Proposition. Suppose Δ is odd and at least 3. Among connected n-vertex graphs with maximum degree Δ, the maximum total interval number exceeds $[\Delta + 1/(2.5\Delta + 7.7)]n/2$ for infinitely many n.

Proof. We prove the claim by using copies of a triangle-free graph H with degree sequence $(\Delta, \ldots, \Delta, \Delta - 1)$ to construct a Δ-regular triangle-free connected graph G with n vertices such that $t(G) \geq [1/(2.5\Delta + 7.7)]n/2$. Begin with a caterpillar C consisting of a path with $k+2$ vertices and $\Delta - 2$ leaves attached to each interior vertex of the path. For each of the $k(\Delta-2)+2$ leaves of C, we provide a copy of H and identify its vertex of degree $\Delta - 1$ with that leaf.

The resulting graph G is triangle-free and Δ-regular, so $I(G) = \Delta n/2 + t(G)$. Because each edge of C is a cut-edge of G, every trail cover of G has an endpoint in each copy of H. Hence $t(G) \geq [k(\Delta - 2) + 2]/2$, and in fact $t(G) = \lceil [k(\Delta - 2) + 2]/2 \rceil$. If H has n' vertices, then $n = [k(\Delta - 2) + 2]n' + k$. We obtain $n < 2t[n' + 1/(\Delta - 2)]$, and hence $t(G) > 1/(n/2)/[n' + 1/(\Delta - 2)]$.

It remains to construct a suitable H with n' as small as possible. When $\Delta = 3$, we form H by subdividing one edge of $K_{3,3}$; here $n' = 7$ and $t(G) > \frac{1}{8}(n/2)$. For larger Δ, consider the lexicographic composition $F_s = C_5[\bar{K}_s]$, expanding each vertex of a 5-cycle into an independent set of size s. The graph F_s is $2s$-regular and triangle-free, and for $s \geq 2$ it has a pair of easily described edge-disjoint Hamiltonian cycles. When $(\Delta + 1)/2$ is odd, we form H by deleting from $F_{(\Delta+1)/2}$ the odd-indexed edges on one Hamiltonian cycle. Since $n' = 2.5(\Delta + 1)$ is odd, this reduces one vertex degree from $\Delta + 1$ to $\Delta - 1$ and the others to Δ. When $(\Delta + 1)/2$ is even, we form H by deleting from $F_{(\Delta+3)/2}$ the odd-indexed edges on one Hamiltonian cycle and all edges on another Hamiltonian cycle. Since $n' = 2.5(\Delta + 3)$ is odd, we again obtain the desired degree sequence. Here $\Delta \geq 7$, which yields the 7.7 in the statement of the result. ∎

THE MAIN RESULT

We consider an n-vertex graph G with maximum degree Δ and wish to prove that $I(G) \leq (\Delta + 1/\Delta)n/2$. We may assume that G has no isolated vertices, because such vertices require no intervals; when $f(v) = \emptyset$ and the intersection graph is taken, v becomes an isolated vertex.

Our approach is to select a set of edge-disjoint trails $T_1, \ldots, T_k$ to cover $E(G)$, in a greedy manner subject to various conditions; each new trail contains some previously uncovered edge. We partition $E(G)$ into sets associated with the trails; the set E_i associated with T_i consists of $E(T_i)$ and the newly covered edges that do not belong to later trails. We will also associate a set $S_i \subseteq V(G)$ with each trail (the union of closed neighborhoods of certain vertices of the trail); these sets will be pairwise disjoint. We can represent E_i using $|E_i| + 1$ intervals, fewer if E_i contains a triangle with at most one edge on T_i.

When G has maximum degree Δ, we have $e(G) \leq \Delta n/2$. If $e(G) = \Delta n/2 - k$, we will use $|E_i| + 1$ intervals for at most $k + n/(2\Delta)$ trails T_i. We do this by ensuring that we use an extra interval for T_i only when there exists $\alpha_i \in \{0, 1, 2\}$ such that $|S_i| \geq (2 - \alpha_i)\Delta$ and the degrees of two new vertices of T_i sum to at most $2\Delta - \alpha_i$. If the values of α_i over the s trails using an extra interval sum to r, then we have $I(G) \leq e(G) + s$ and $e(G) \leq \Delta n/2 - r/2$. Hence $I(G) \leq \Delta n/2 + (2s - r)/2$. Also we have associated $(2s - r)\Delta$ vertices with these trails. Since $(2s - r)\Delta \leq n$, we have the desired bound. The remainder of the proof consists of showing that we can choose the trails to ensure these conditions.

We use ''open'' trails; these are trails with two distinct endpoints. We say that a trail is *closable* if its endpoints are adjacent via an edge not belonging to the trail. When T is closable, we let T' denote the closed trail formed by adding the edge between the endpoints of T.

Theorem. Every simple graph with n vertices and maximum degree Δ has total interval number at most $(\Delta + 1/\Delta)n/2$. Furthermore, equality holds only when every component is $K_{\Delta,\Delta}$.

Proof. We select a sequence of pairwise edge-disjoint open trails $T_1, \ldots, T_k$ in a greedy manner. The *new vertices* of T_i are the vertices in T_i that do not appear in $T_1, \ldots, T_{i-1}$ and that cover at least one edge not covered by vertices of earlier trails. The *new edges* of T_i are the previously uncovered edges that are covered by new vertices of T_i. Note that an edge of T_i is new (for T_i) if and only if both its endpoints are new.

We choose each T_i to be an open trail having endpoints that are new. Among these, we choose T_i with maximum number of new vertices. Among these, we choose T_i to be closable if such a candidate is available. Among the remaining candidates for T_i, we choose T_i with minimum length.

The sequence ends when all edges are covered. The set E_i of edges associated with T_i is $E(T_i)$ together with the new edges that do not belong to later trails. By construction, these edge sets are disjoint. We postpone the definition of the vertex set S_i associated with T_i.

Claim 1. If T_i is not closable and has endpoint v, then only one edge incident to v belongs to T_i. Otherwise, we delete the initial portion of T_i up to the next appearance of a new vertex other than the other endpoint of T_i (this may be v again). The shorter trail T is open and has the same new set as T_i; it may be closable, but since T_i is not closable, we would in either case choose T in preference to T_i.

Claim 2. If the vertices of T_i are not all new, then T_i is not closable and the end edges of T_i are new. Let x be the first vertex of T_i that is not new, belonging to an earlier trail T_j. If T_i is closable, then T_j can absorb the closed trail T_i' to enlarge its new set. If T_i is not closable, then the first edge of T_i is new unless it is ux. By the maximality of the new set, every neighbor of u along a new edge belongs to T_i. If v is the first such vertex on T_i, then the u, v-portion of T_i together with the edge uv forms a closed trail containing x that can be absorbed by T_j to enlarge its new set.

Claim 3. If T_i is closable, then no vertex of T_i appears in another trail or has a neighbor in a later trail. By Claim 2, every vertex of T_i is new. If w is a vertex of T_i that equals or is adjacent to a vertex w' of a later trail T_j, then we can traverse T_i' starting at w, enter T_j at w', and continue to an end of T_j, replacing T_i by a trail with at least two more new vertices.

Claim 4. If T_i is not closable, then there is no edge to a later trail from an endpoint of T_i or from its neighbor along T_i. Any such edge permits an extension of T_i (or of T_i minus its endpoint) using a portion of T_j that has at least two new vertices, thereby creating a trail with more new vertices than T_i.

The *start vertices* of T_i are its endpoints if T_i is not closable, or all of its vertices if T_i is closable. By Claim 3, every start vertex of T_i is a new vertex of T_i.

Claim 5. No vertex of T_i is adjacent to two start vertices of later trails, or to a start vertex of T_i and a start vertex of a later trail. Suppose $w \in V(T_i)$ has neighbors x, y that are start vertices of T_j, T_k, respectively, with $i \le j \le k$ and $i \neq k$. By Claim 3, T_i is not closable. By the ''newness'' of start vertices (and by Claim 4 if $i = j$), wx, wy do not belong to $E(T_i)$. If $j = k, T_i$ could thus absorb a portion of T_j that contains a new vertex, giving T_i more new vertices. Hence we may assume $i \le j < k$. In this case, $wy \notin E(T_j)$, since y is new in T_k. If T_j is closable, then $j > i$ and Claim 3 yields $wx \notin E(T_j)$. If T_j is not closable, then Claim 1, Claim 4 and the edge wy imply that $wx \notin E(T_j)$. Now T_j, which we can view as ending at y, can be extended via w to absorb at least two new vertices from T_k.

Claim 6. If u, v are start vertices of T_i, T_j with $i < j$, then u, v are nonadjacent and have no common neighbor. By Claims 3 and 4, a start vertex of T_i has no neighbor in a later trail. No start vertex of T_i has a neighbor outside all trails, because such a neighbor could be used to enlarge the new set of T_i. By Claim 5, u and v have no common neighbor in trail T_i or earlier.

Claim 7. If E_i contains a triangle with at most one edge on T_i, then E_i can be represented using $|E_i|$ intervals. If the vertices of T_i are $v_1, \ldots, v_n$ in order (with repetition), then we represent T_i by assigning the interval $(j - \frac{2}{3}, j + \frac{2}{3})$ to v_j. This uses $e(T_i) + 1$ intervals. For each additional edge $e \in E_i$ that is not in the triangle, suppose $e = xv_j$ where v_j is a new vertex of T_i. We represent e by adding a small interval for x within $(j - \frac{1}{3}, j + \frac{1}{3})$ (intersecting only the interval for v_j). If the triangle in E_i contains an edge $v_j v_{j+1}$ of T_i, then we add an interval for their common neighbor in E_i within $(j + \frac{1}{3}, j + \frac{2}{3})$, gaining two edges for one interval. If it contains no edge of T_i, we select some $v_j \in V(T_i)$ on the triangle and add a common interval for the other two vertices of the triangle within $(j - \frac{1}{3}, j + \frac{1}{3})$, gaining three edges for two intervals. In total, we have used $|E_i|$ intervals.

Having proved these claims, we let S_i consist of the start vertices of T_i and their neighbors. By Claim 6, the sets S_i are pairwise disjoint. Choose two start vertices u, v in T_i with the minimum degree sum. If $d(u) + d(v) \leq 2\Delta - 2$, then by the discussion before the theorem statement we do not need to save an interval for T_i. If $d(u) + d(v) = 2\Delta - 1$, then one of u, v has degree Δ and we have $|S_i| > \Delta$.

Hence we may assume that every start vertex of T_i has degree Δ. By the computation before the theorem statement, it remains only to show that $|S_i| \geq 2\Delta$ if E_i does not contain a triangle with at most one edge on T_i. If some pair of start vertices on T_i has no common neighbor, then $|S_i| \geq 2\Delta$, so we may assume that every pair of start vertices has a common neighbor.

Suppose first that T_i is not closable. Let u, v be the endpoints of T_i, and let w be a common neighbor; by Claim 5, w does not belong to an earlier trail. If neither of $\{uw, vw\}$ belongs to T_i, then T_i can be extended by uw to obtain a closable trail with the same new set as T_i, which would be preferred to T_i. This includes the case where u, v are adjacent and T_i has length 1. In the remaining case, u, v are nonadjacent and any common neighbor of them is adjacent to one of them using an end edge of T_i. There are at most two such common neighbors. Hence $|S_i| \geq 2 + 2\Delta - 2 = 2\Delta$, as desired.

Finally, suppose that T_i is closable, which requires at least three vertices, each pair of which has a common neighbor. By Claims 3 and 5, the common neighbors of vertices in T_i also lie in T_i. Furthermore, every edge of T_i' forms a triangle only using two other edges of T_i'; otherwise, we can use the endpoints of that edge as the endpoints of T_i and use the common neighbor to form a triangle having at most one edge on T_i.

Since T_i' forms a connected subgraph of G, it has a vertex w that is not a cut-vertex of T_i'. Deleting from T_i' any set of edges incident to w leaves a connected subgraph, except possibly for isolating w. Every edge wv incident to w in T_i' lies on a triangle in T_i'; let u be a third vertex of this triangle. Deleting $\{wv, wu\}$ from T_i' leaves a subgraph having a u, v-Eulerian trail T. Now u, v, w form a triangle with only one edge on T. Furthermore, every edge of E_i is incident to at least one vertex of T, because when T_i is closable every edge of E_i has both endpoints on T_i. By the construction in Claim 7, we can represent E_i using only $|E_i|$ intervals, saving one for the edges $\{wv, wu\}$.

We have resolved all cases, and the proof of the bound is complete. Next we consider how equality may be achieved. We may assume that G is connected. It suffices to show that if $G \neq K_{\Delta,\Delta}$, then we save an extra interval for T_1.

If T_1 is closable, then Claim 3 implies that $V(T_1) = V(G)$, and hence $E_1 = E(G)$. If G is not Δ-regular or if $n > 2\Delta$, then $|E(G)| + 1 < (\Delta + 1/\Delta)n/2$, and we are done. If G is Δ-regular and $\Delta \geq n/2$, then G is Hamiltonian, by Dirac's Theorem. If $G \neq K_{\Delta,\Delta}$, then G has a triangle, by Turán's Theorem. By the choice of T_1 to minimize length, T_1' is a Hamiltonian cycle. If T_1' uses any edge of the triangle, then we delete that edge from T_1' to obtain T_1. Hence we can choose T_1 so that we have a triangle with at most one edge on T_1. By Claim 7, we can now represent $E(G)$ using $|E(G)| = \Delta n/2$ edges.

If T_1 is not closable, recall the computation of our bound on $I(G)$. We have $I(G) \leq \Delta n/2 + (2s - r)/2$, where there are s trails T_i using $|E_i| + 1$ intervals and $r = \sum \alpha_i$, with $2\Delta - \alpha_i$ being the sum of the degrees of the chosen vertices on T_i. We proved the bound by associating $(2s - r)\Delta$ vertices with these trails, since then $(2s - r)\Delta \leq n$. We have strict inequality if some T_i uses only $|E_i|$ intervals (since its start vertices are not in the sets S_j associated with other trails) or if for some T_i the associated set S_i has more than $2\Delta - \alpha_i$ vertices.

Now consider T_1, with endpoints u, v. If $d(u) + d(v) < 2\Delta$, then $|S_i| > (2 - \alpha_i)\Delta$ immediately, so we may assume $d(u) + d(v) = 2\Delta$. By Claim 4, all neighbors of v lie in T_1. By Claim 1, only one edge incident to v belongs to T_1. Since our greedy selection prefers closable trails, this implies that u, v are not adjacent (unless $uv = E(T_1)$, in which case $\Delta = 1$ and $G = K_{\Delta,\Delta}$). Suppose the vertices of T_1 are $u = x_1, x_2, \ldots, x_m = v$ in order. If v is adjacent to both x_i and x_{i+1} for some $i < m - 2$, then it forms a triangle with one edge on T_i, and Claim 7 applies. If $i = m - 2$, then because v has no neighbors on later trails, we can again save an interval for this triangle. Hence we may assume that v does not have consecutive neighbors on T_i. If v is adjacent to x_i for $i < m - 1$, then x_{i+1} has no neighbor w in another trail T_j. Otherwise, we could follow $x_1, \ldots, x_i, v, x_{m-1}, \ldots, x_{i+1}, w$ and continue in T_j to enlarge the new set of T_1. Since the successors on T_1 of neighbors of v have no neighbors in later trails, they appear in no later S_j, and we can add them to S_1. Now S_1 consists of at least the neighbors of v, their successors on T_1, and u, which totals $2\Delta + 1$ vertices. ∎

The alterations that are used to improve trails always increase the new set or decrease the length (or make it closed); there can be at most n^2 of these changes for each trail. Also the search for whether a change is needed takes polynomial time. Hence this proof can be implemented as a polynomial algorithm to produce a representation that satisfies the bound.

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