An Intersection Theorem for Systems of Sets

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ABSTRACT

Erdős and Rado defined a Δ-system, as a family in which every two members have the same intersection. Here we obtain a new upper bound on the maximum cardinality $\varphi(n, q)$ of an n-uniform family not containing any Δ-system of cardinality q. Namely, we prove that, for any $\alpha > 1$ and q, there exists $C = C(\alpha, q)$ such that, for any n,

$$\varphi(n, q) \le Cn!\left(\frac{(\log\log\log n)^2}{\alpha \log\log n}\right)^n .$$

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1. INTRODUCTION

Erdős and Rado [3] introduced the notion of a Δ-system. They called a family $\mathscr{H}$ of finite sets a *Δ-system* if every two members of $\mathscr{H}$ have the same intersection.

Let $\varphi(n, q)$ (respectively, $\varphi(n, q, p)$) denote the maximum cardinality of an n-uniform family not containing any Δ-system of cardinality q (respectively, n-uniform family not containing any Δ-system of cardinality q such that there are no p pairwise disjoint sets).

Erdős and Rado [3] proved that

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$$(q-1)^n \leq \varphi(n,q) \leq (q-1)^n n! \left\{ 1 - \sum_{t=1}^{n-1} \frac{t}{(t+1)!(q-1)^t} \right\} \tag{1}$$

and conjectured that

$$\varphi(n,3) < c^n \qquad \text{for some absolute constant } c\,.$$

Abbott, Hanson, and Sauer [1] improved the upper bound in (1) down to

$$\varphi(n,q) \leq (n+1)! \left\{ \frac{q-1+(q^2+6q-7)^{0.5}}{4} \right\}^n .$$

We are interested in asymptotic bounds on $\varphi(n,q)$ for fixed q. The best published upper bound of this kind is due to Spencer [5]: For fixed q, $\epsilon > 0$, there exists $C = C(q,\epsilon)$ such that

$$\varphi(n,q) < C(1+\epsilon)^n n!\,,$$

and

$$\varphi(n,3) < e^{cn^{3/4}} n!\,.$$

Füredi and Kahn (see [2]) proved that

$$\varphi(n,3) < e^{c\sqrt{n}} n!\,.$$

In [4], it was proved that, for any integer $\alpha > 1$, there exists $C = C(\alpha)$ such that, for any n,

$$\varphi(n,3) \leq C\, n! \alpha^{-n}\,.$$

It appeared that the bound can be extended from $q = 3$ to any fixed q (using ideas of Spencer [5]). The aim of the present paper is to prove:

Theorem 1. *Let $n \geq 1$ and $q \geq p \geq 2$ be integer and $\alpha(n,q) = \max\{20, \frac{\log\log n}{20q \log\log\log n}\}$. Then there exists $C(q)$ such that for all n,*

$$\varphi(n,q,p) \leq C(q) n! \lfloor \log\log n \rfloor^{4p\alpha(n,q)} \alpha(n,q)^{-n}\,.$$

Maybe more visual is the following immediate consequence of Theorem 1.

Corollary 2. *For each integers $q > 2$ and $\alpha > 1$, there exists $D(q,\alpha)$ such that, for all n,*

$$\varphi(n,q) \leq D(q,\alpha) n! \left(\frac{(\log\log\log n)^2}{\alpha \log\log n} \right)^n .$$

To derive Corollary 2 from Theorem 1, consider arbitrary integers $q > 2$ and $\alpha > 1$. Let $n_0 = \min\{\,|\log\log\log n > 400q\alpha\}$. Then $\alpha(n,q) = \frac{\log\log n}{20q \log\log\log n}$, and, by Theorem 1,

$$\begin{aligned}\varphi(n,q) &= \varphi(n,q,q) \leq C(q) n! (\log\log n)^{\frac{\log\log n}{5 \log\log\log n}} \left(\frac{20q \log\log\log n}{\log\log n} \right)^n \\ &< C(q) n! (\log n)^{1/5} \left(\frac{(\log\log\log n)^2}{2\alpha \log\log n} \right)^n \leq C(q) n! \left(\frac{(\log\log\log n)^2}{\alpha \log\log n} \right)^n .\end{aligned}$$

Choosing $D(n, \alpha) = \max\{C(q), \varphi(n_0, q)\}$, we get Corollary 2.

The proof of Theorem 1 almost completely repeats that in [4]. In particular, in the course of proofs some inequalities are true when n is large in comparison with q. We choose $C(q)$ so that the statement of the theorem holds for smaller n.

All the logarithms throughout the paper are taken to the base e.

2. PRELIMINARY LEMMAS

Call a family $\mathcal{F}$ of sets a *(q, n, k, p)-family* if it is an n-uniform family not containing any Δ-system of cardinality q such that the cardinality of the intersection of each two members of $\mathcal{F}$ is at most $n-k$ and there are no p pairwise disjoint members of $\mathcal{F}$.

Lemma 1. *For any (q, n, k, q)-family $\mathcal{F}$,*

$$|\mathcal{F}| \le (q-1)^{n-k+1} \frac{n!}{k!}.$$

Proof. We use induction on $n-k$. Any (q, k, k, q)-family has at most $q-1$ members. Hence the lemma is true for $n-k=0$.

Let the lemma be valid for $n-k \le m-1$ and $\mathcal{F}$ be a $(q, m+k, k, q)$-family. Choose $q-1$ members $A_1, \dots, A_{q-1}$ of $\mathcal{F}$ with maximum cardinality of their union and let $Z = \bigcup_{i=1}^{q-1} A_i$. Then each $A \in \mathcal{F}$ has a nonempty intersection with Z.

For any $x \in Z$, let $\mathcal{F}(x) = \{A \in \mathcal{F} \mid x \in A\}$, $\tilde{\mathcal{F}}(x) = \{A \setminus \{x\} \mid A \in \mathcal{F}(x)\}$. Then, for all $x \in Z$, $\tilde{\mathcal{F}}(x)$ is a $(q, m+k-1, k, q)$-family. Thus,

$$|\mathcal{F}| \le \sum_{x \in Z} |\tilde{\mathcal{F}}(x)| \le |Z|(q-1)^m \frac{(m-1+k)!}{k!} \le (q-1)^{m+1} \frac{(m+k)!}{k!}. \qquad \square$$

From now on, we suppose that for each $m \le n-1$ and $1 < p' < p$,

$$\varphi(m, q) \le C(q) m! \lfloor \log\log m \rfloor^{4q\alpha(m,q)} \alpha(m, q)^{-m}, \tag{2}$$

$$\varphi(n, q, p') \le C(q) n! \lfloor \log\log n \rfloor^{4p'\alpha(n,q)} \alpha(n, q)^{-n}. \tag{3}$$

The following observation from [5] will be used throughout the paper. Let $B_1, \dots, B_t$ be pairwise disjoint finite sets and $\mathcal{F}$ be a $(q, n, 1, q)$-family such that $|A \cap B_i| \ge b_i$ for each $A \in \mathcal{F}$. Then

$$|\mathcal{F}| \le \binom{|B_1|}{b_1} \cdot \dots \cdot \binom{|B_t|}{b_t} \varphi(n - b_1 - \dots - b_t, q). \tag{4}$$

Lemma 2. *Let $0 < r \le k \le n/2$ and for any members $A_1, \dots, A_r$ of a $(q, n, 1, q)$-family $\mathcal{F}$,*

$$|A_1 \cup \dots \cup A_r| \le rn - kr^2/2. \tag{5}$$

Then

$$|\mathcal{F}| \le C(q) \frac{n!}{k!}.$$

Proof. For $r=1$ the lemma is valid since (5) is impossible as $r=1$. Suppose that the lemma is true for $r \le s-1$ and $|\mathcal{F}| > C(q)n!/k!$. By the induction hypothesis, there exist $A_1, \ldots, A_{s-1} \in \mathcal{F}$ such that for the set $B = A_1 \cup \cdots \cup A_{s-1}$, we have $|B| > (s-1)n - k(s-1)^2/2$. If the lemma does not hold for $\mathcal{F}$, then, for all $A \in \mathcal{F}$,

$$|A \cap B| > b + ((s-1)n - k(s-1)^2/2) - (sn - ks^2/2) = k(s - 1/2)\,,$$

and there is an i, $1 \le i \le s-1$ such that $|A \cap A_i| > k$. Thus, by (4),

$$\begin{aligned} |\mathcal{F}| &\le (s-1)\binom{n}{k+1}\varphi(n-k-1, q) \\ &\le (s-1)\binom{n}{k+1}\lfloor \log\log(n-k-1)\rfloor^{4q\alpha(n-k-1,q)} \\ &\quad \times C(q)(n-k-1)!\alpha(n-k-1,q)^{-n+k+1} \\ &\le C(q)n!\alpha(n-k-1,q)^{-n+k+1}\lfloor \log\log(n-k-1)\rfloor^{4q\alpha(n-k-1,q)}\frac{s-1}{(k+1)!}\,. \end{aligned}$$

Note that for sufficiently large $n-k-1$, we have $\lfloor \log\log(n-k-1)\rfloor^{4q\alpha(n-k-1,q)} \le (\log(n-k-1))^{1/5}$ and hence $|\mathcal{F}| < C(q)n!/k!$. □

Lemma 3. *Let $\xi \ge 2$, $1 \le t < s \le n$ and $\mathcal{F}$ be a $(q,s,1,q)$-family with $|\mathcal{F}| \ge Cs!\xi^{-s}$. Then there exist $\mathcal{F}' \subset \mathcal{F}$ and X such that*

(1) $|X| = s-t$,
(2) for all $A \in \mathcal{F}'$, $A \supset X$;
(3) $|\mathcal{F}'| \ge Ct!\beta^{-t}$, *where* $\beta = (2q\xi)^{s/t}$.

Proof. Case 1. *For any $A \in \mathcal{F}$, $|\{B \in \mathcal{F} \mid |B \cap A| \ge s-t\}| \le Ct!2^s\beta^{-t} - 1$.* Let $\mathcal{H}$ be a maximal $(q,s,t+1,q)$-subfamily of $\mathcal{F}$. By the choice of $\mathcal{H}$, for each $B \in \mathcal{F}\backslash\mathcal{H}$, there is an $A \in \mathcal{H}$ with $|B \cap A| \ge s-t$. Thus, $|\mathcal{F}\backslash\mathcal{H}| \le |\mathcal{H}|(Ct!2^s\beta^{-t} - 1)$, and

$$|\mathcal{H}| \ge \frac{|\mathcal{F}|}{Ct!2^s\beta^{-t}} = \frac{s!(2q\xi)^s}{t!(2\xi)^s} = q^s\frac{s!}{t!}\,.$$

But the existence of such a big $(q,s,t+1,q)$-family contradicts Lemma 1.

Case 2. *There exists $A \in \mathcal{F}$ such that $|\{B \in \mathcal{F} \mid |B \cap A| \ge s-t\}| \ge \lfloor Ct!2^s\beta^{-t}\rfloor$.* If $\lfloor Ct!2^s\beta^{-t}\rfloor \le 1$ then the statement is trivial. Otherwise, for some $X \subset A$ with $|X| = s-t$,

$$|\{B \in \mathcal{F} \mid B \cap A \supset X\}| \ge \lfloor Ct!2^s\beta^{-t}\rfloor\binom{s}{s-t}^{-1} > Ct!\beta^{-t}\,.$$

This is the family we need. □

3. MAIN CONSTRUCTION

Let $\mathcal{F}$ be a $(q, n, 1, p)$-family with $|\mathcal{F}| = \varphi(n, q, p)$. Assume that

$$|\mathcal{F}| > C(q)n! \lfloor \log\log n \rfloor^{4p\alpha(n,q)} \alpha(n, q)^{-n} .$$

The idea is to find a (not too large) family of collections of pairwise disjoint (and considerably small) sets such that most members of $\mathcal{F}$ intersect each set from at least one of these collections and then apply (4). We put

$$\alpha = \lfloor \alpha(n, q) \rfloor , \qquad y = \lfloor n/3\alpha \rfloor , \qquad m = 3\alpha - 1 ,$$

$$k = \lceil \tfrac{n}{\log n} (\log\log n)^3 \rceil , \qquad r = \lfloor \log\log n \rfloor .$$

Lemma 4. *For all $s = 0, 1, \ldots, m$ and for $i_0 = 1$ and any $i_1, \ldots, i_s \in \{1, \ldots, r\}$ there are subfamilies $\mathcal{F}(1, i_1, \ldots, i_s)$ of the family $\mathcal{F}$ and sets $X(i_1, \ldots, i_s)$ and $Z(1, i_1, \ldots, i_{s-1})$ such that, for $s \neq 0$ and for any $i_1, \ldots, i_s, i'_s \in \{1, \ldots, r\}$,*

(i) $\mathcal{F}(1, i_1, \ldots, i_s) \subset \mathcal{F}(1, i_1, \ldots, i_{s-1})$,
(ii) *for all* $A \in \mathcal{F}(1, i_1, \ldots, i_s)$, $A \supset X(i_1) \cup X(i_1, i_2) \cup \cdots \cup X(i_1, i_2, \ldots, i_s)$,
(iii) *the sets* $X(i_1)$, $X(i_1, i_2), \ldots, X(i_1, i_2, \ldots, i_s)$ *are pairwise disjoint,*
(iv) $|X(i_1, i_2, \ldots, i_s)| = y$,
(v) $|Z(1, i_1, i_2, \ldots, i_{s-1})| \leq kr(r+1)/2$,
(vi) $X(i_1, i_2, \ldots, i_{s-1}, i_s) \cap X(i_1, i_2, \ldots, i_{s-1}, i'_s) \subset Z(1, i_1, i_2, \ldots, i_{s-1})$,
(viii) $|\mathcal{F}(1, i_1, \ldots, i_s)| \geq C(n - sy)! \xi_s^{sy-n}$, *where*

$$\xi_s = ((2\alpha)^{\frac{n}{n-sy}} (4q)^{\frac{n+(n-y)+\cdots+(n-(s-1)y)}{n-sy}}) = ((2\alpha)^n (4q)^{ns-s(s-1)y/2})^{\frac{1}{n-sy}} .$$

Proof. We use induction on s. Put $\mathcal{F}(1) := \mathcal{F}$, $\xi_0 := 2\alpha$.

Steps ($0 \leq s < m$). We have at hand $\mathcal{F}(1, i_1, \ldots, i_s)$ for any $i_1, \ldots, i_s \in \{1, \ldots, r\}$ and if $s > 0$ we also have sets $X(i_1, \ldots, i_s)$ and $Z(1, i_1, \ldots, i_{s-1})$ as needed. Consider

$$\begin{aligned} \tilde{\mathcal{F}} &= \tilde{\mathcal{F}}(1, i_1, \ldots, i_s) \\ &= \{A \backslash (X(i_1) \cup X(i_1, i_2) \cup \cdots \cup X(i_1, i_2, \ldots, i_s)) \mid A \in \mathcal{F}(1, i_1, \ldots, i_s)\} . \end{aligned}$$

According to the statements of the lemma, $\tilde{\mathcal{F}}$ is a $(q, n - sy, 1, q)$-family. Note that $n - my \geq n - (3\alpha - 1)n/(3\alpha) = n/(3\alpha)$ and hence for each $0 \leq s \leq m$,

$$(ns - s(s-1)y/2)/(n - sy) \leq ns/(n - my) \leq \frac{mm}{n/(3\alpha))} < (3\alpha)^2 .$$

Therefore, for each $0 \leq s \leq m$.

$$\xi_s \leq (2\alpha)^{3\alpha} (4q)^{9\alpha^2} \leq (8q)^{9\alpha^2} \tag{6}$$

and due to Statement (vii) of the lemma, we can use Lemma 3. This Lemma 3 provides that there exists X_1 of cardinality y and $\mathcal{H}_1 \subset \tilde{\mathcal{F}}$ with $|\mathcal{H}_1| \geq C(n - (s + 1)y)! \beta^{(s+1)y-n}$ (where $\beta = (2q\xi_s)^{\frac{n-sy}{n-(s+1)y}}$) such that each $A \in \mathcal{H}_1$ contains X_1. We put $Z_1 := \emptyset$.

Suppose that $(q, n - sy, 1, q)$-families $\mathcal{H}_1, \ldots, \mathcal{H}_l$ and sets $X_1, \ldots, X_l$, $Z_1, \ldots, Z_l$ are constructed and that, for each $1 \leq j \leq l$, $1 \leq j' \leq l$, $j \neq j'$,

- $|X_j| = y$,
- $|Z_l| \leq kl(l-1)/2$,
- $X_j \cap X_{j'} \subset Z_l$, (7)
- for all $A \in \mathcal{H}_j$, $X_j \subset A$,
- $|H_j| \geq C(n-(s+1)y)!\xi_{s+1}^{(s+1)y-n}$.

If $l < r$, then we construct $\mathcal{H}_{l+1}$, X_{l+1}, and Z_{l+1} as follows. Note that, for each $A \in \tilde{\mathcal{F}}$, we have

$$|A| = n - sy \geq 2y > lk + y\ ,$$

and by (4) the number $N(s,l)$ of $A \in \tilde{\mathcal{F}}$ with $|A \cap (X_1 \cup \cdots \cup X_l)| \geq lk$ does not exceed

$$\begin{aligned} &\binom{|X_1 \cup \cdots \cup X_l|}{lk}\varphi(n-sy-lk, q) \\ &\leq \binom{ly}{lk} C(q)(n-sy-lk)!\lfloor \log\log(n-sy-lk)\rfloor^{4q\alpha(n-sy-lk,q)} \\ &\quad \times \alpha(n-sy-lk, q)^{sy+lk-n}\ . \end{aligned}$$

Since $n - sy - lk > y$, for large n we have

$$\begin{aligned} &\lfloor \log\log(n-sy-lk)\rfloor^{4q\alpha(n-sy-lk,q)}\alpha(n-sy-lk,q)^{sy+lk-n} \\ &\quad \leq (\log(n-sy-lk))^{1/5}\alpha(n-sy-lk,q)^{sy+lk-n} < 1\ . \end{aligned}$$

Consequently,

$$N(s,l) \leq \left(\frac{ely}{lk}\right)^{lk} C(q)(n-sy-lk)! \leq \frac{C(q)(n-sy)!}{(k/e)^{lk}}\ .$$

But for large n we have

$$\left(\frac{k}{e}\right)^k \geq \left(\frac{n}{e\log n}\right)^{\frac{n(\log\log n)^3}{\log n}} \geq e^{0.5n(\log\log n)^3} > e^{n(3q\alpha)^3}\ .$$

In view of (6), we obtain

$$k! \geq (k/e)^k \geq (2\xi_m)^n\ , \tag{8}$$

and for the family $\mathcal{H}' := \{A \in \tilde{\mathcal{F}} \mid |A \cap (X_1 \cup \cdots \cup X_l)| < lk\}$ we have $|\mathcal{H}'| \geq |\tilde{\mathcal{F}}| - C(q)(n-sy)!(2\xi_s)^{sy-n} \geq C(q)(n-sy)!(2\xi_s)^{sy-n}$. Then by Lemma 3 there exist $\mathcal{H}_{l+1} \subset \mathcal{H}'$ and X_{l+1} with $|X_{l+1}| = y$ such that each $A \in \mathcal{H}_{l+1}$ contains X_{l+1} and $|\mathcal{H}_{l+1}| \geq C(q)(n-(s+1)y)!\beta^{(s+1)y-n}$, where

$$\begin{aligned} \beta &= (2q \times 2\xi_s)^{\frac{n-sy}{n-(s+1)y}} = (4q \times ((2\alpha)^n(4q)^{ns-s(s-1)y/2})^{\frac{1}{n-sy}})^{\frac{n-sy}{n-(s+1)y}} \\ &= ((2\alpha)^n(4q)^{n(s+1)-s(s+1)y/2})^{\frac{1}{n-(s+1)y}} = \xi_{s+1}\ . \end{aligned}$$

By definition of $\mathcal{H}'$,

$$|X_{l+1} \cap (X_1 \cup \cdots \cup X_l)| < lk .$$

Putting $Z_{l+1} := Z_l \cup (X_{l+1} \cap (X_1 \cup \cdots \cup X_l))$, we have $|Z_{l+1}| \le |Z_l| + lk \le kl(l+1)/2$ and conditions (7) are fulfilled for $l+1$. Thus we can proceed till $l = r$.

After constructing $\mathcal{H}_r$, X_r, and Z_r, we put, for $j = 1, \ldots, r$, $X(i_1, i_2, \ldots, i_s, j) := X_j$ and

$$\mathcal{F}(1, i_1, \ldots, i_s, j) := \{A \cup X(i_1) \cup X(i_1, i_2) \cup \cdots \cup X(i_1, i_2, \ldots, i_s, j) \mid A \in \mathcal{H}_i\} ,$$

and

$$Z(1, i_1, \ldots, i_s) := Z_r .$$

By construction, the statements (i)–(vii) of the lemma will be fulfilled for $s+1$ □

Lemma 5. *For all $s = 0, 1, \ldots, m+1$ and for all $i_1, \ldots, i_s \in \{1, \ldots, r\}$, there are sets $X(i_1, \ldots, i_s)$ and $Z(1, i_1, \ldots, i_{s-1})$ and for all $i_1, \ldots, i_{m+1} \in \{1, \ldots, r\}$, there are sets $A(i_1, \ldots, i_{m+1}) \in \mathcal{F}$ such that*

- *(i) the sets $X(i_1), X(i_1, i_2), \ldots, X(i_1, i_2, \ldots, i_{m+1})$ are pairwise disjoint;*
- *(ii) $|X(i_1, i_2, \ldots, i_s)| = y$ if $1 \le s \le m$,*
- *(iii) $|X(i_1, i_2, \ldots, i_{m+1})| = n - my$,*
- *(iv) $A(i_1, \ldots, i_{m+1}) = X(i_1) \cup X(i_1, i_2) \cup \cdots \cup X(i_1, i_2, \ldots, i_{m+1})$, for all $s = 1, \ldots, m$ and for all $i_1, \ldots, i_s, i'_s \in \{1, \ldots, r\}$,*
- *(v) $X(i_1, i_2, \ldots, i_{s-1}, i_s) \cap X(i_1, i_2, \ldots, i_{s-1}, i'_s) \subset Z(1, i_1, i_2, \ldots, i_{s-1})$,*
- *(vi) $|Z(1, i_1, i_2, \ldots, i_{s-1})| \le k(r+1)/2$.*

Proof. For $s = 0, 1, \ldots, m$ and for any $i_1, \ldots, i_s \in \{1, \ldots, r\}$, consider $\mathcal{F}(1, i_1, \ldots, i_s)$, $X(i_1, \ldots, i_s)$, and $Z(1, i_1, \ldots, i_{s-1})$ from Lemma 4.

Now, for an arbitrary $(m+1)$-tuple $(1, i_1, \ldots, i_m)$, consider

$$\begin{aligned}\mathcal{H} &= \mathcal{H}(1, i_1, \ldots, i_m) \\ &:= \{A \setminus (X(i_1) \cup X(i_1, i_2) \cup \cdots \cup X(i_1, i_2, \ldots, i_m)) \mid A \in \mathcal{F}(1, i_1, \ldots, i_m)\} .\end{aligned}$$

By construction, $\mathcal{H}$ is a $(q, n - my, 1, q)$-family and, by Lemma 4, $|\mathcal{H}| \ge C(n - my)! \xi_m^{my-n}$.

Recall that $n - my \ge n/(3\alpha)$. By (8), for large n, $|\mathcal{H}| > C(q)(n - my)!/k!$ and by Lemma 2 [note that $0 < r < k < (n - my)/2$], there exist $A_1, \ldots, A_r \in \mathcal{H}$ such that

$$|A_1 \cup \cdots \cup A_r| > r(n - my) - kr^2/2 . \tag{9}$$

Let

$$Z(1, i_1, \ldots, i_m) := \bigcup_{1 \le j < h \le r} A_j \cap A_h ,$$

and for $j = 1, \ldots, r$, $X(i_1, i_2, \ldots, i_m, j) := A_j$, and

$$A(i_1, \ldots, i_m, j) = X(i_1) \cup X(i_1, i_2) \cup \cdots \cup X(i_1, i_2, \ldots, i_m) \cup A_j .$$

In view of (9), $|Z(1, i_1, \ldots, i_m)| \le kr^2/2$. Now, by Lemma 4 and the construction, all the statements of the lemma are fulfilled. □

Lemma 6. *If* $A \in \mathcal{F}$ *and* $A \cap A(i_1, \ldots, i_{m+1}) \neq \emptyset$ *for all* $i_1, \ldots, i_{m+1} \in \{1, \ldots, r\}$, *then there exist* s, $0 \leq s \leq m$ *and* $i_1, \ldots, i_s \in \{1, \ldots, r\}$ *such that*

$$A \cap X(i_1, i_2, \ldots, i_s, j) \neq \emptyset \qquad \forall j \in \{1, \ldots, r\} . \tag{10}$$

Proof. Assume that for some $B \in \mathcal{F}$ for each $s, 0 \leq s \leq m$ and each $i_1, \ldots, i_s \in \{1, \ldots, r\}$, there exists $j^*(1, \ldots, i_s)$ such that

$$B \cap X(i_1, i_2, \ldots, i_s, j^*(1, \ldots, i_s)) = \emptyset .$$

Let further $q_0 = 1$ and for $s = 1, \ldots, m+1$, $q_s = j^*(q_0, \ldots, q_{s-1})$. Then B has empty intersection with every member of the sequence $X(q_1), X(q_1, q_2), \ldots, X(q_1, q_2, \ldots, q_{m+1})$. But this means that B is disjoint from $A(q_1, q_2, \ldots, q_{m+1})$, and we are done □

Completion of the Proof of the Theorem. Consider

$$Z := \bigcup_{s=1}^{m+1} \bigcup_{(1, i_1, \ldots, i_{s-1})} Z(1, i_1, \ldots, i_{s-1}) .$$

Clearly, $|Z| \leq (1 + r + r^2 + \cdots + r^m) kr(r+1)/2 \leq kr^{m+2} = kr^{3\alpha+1}$.

Denote by $\mathcal{B}(i_1, \ldots, i_{m+1})$ the collection $\{A \in \mathcal{F} \mid A \cap 4(i_1, \ldots, i_{m+1}) = \emptyset\}$.

Let $\mathcal{E} = \{A \in \mathcal{F} \mid A \cap Z \neq \emptyset\}$ and

$$\mathcal{H}(1, i_1, \ldots, i_s) = \{A \in \mathcal{F} \backslash \mathcal{E} \mid A \cap X(i_1, i_2, \ldots, i_s, j) \neq \emptyset \ \forall j \in \{1, \ldots, r\}\} .$$

By Lemma 5, for each s, $0 \leq s \leq m$ and $i_1, \ldots, i_s \in \{1, \ldots, r\}$, the sets $X(i_1, i_2, \ldots, i_s, j) \backslash Z$ for distinct j are pairwise disjoint. Hence by Lemma 6, we can write $\mathcal{F}$ in the form

$$\mathcal{F} = \bigcup_{(i_1, \ldots, i_{m+1})} \mathcal{B}(i_1, \ldots, i_{m+1}) \cup \mathcal{E} \cup \left(\bigcup_{s=0}^{m} \bigcup_{(1, i_1, \ldots, i_s)} \mathcal{H}(1, i_1, \ldots, i_s) \right) .$$

By (3),

$$\left| \bigcup_{(i_1, \ldots, i_{m+1})} \mathcal{B}(i_1, \ldots, i_{m+1}) \right| \leq r^{m+1} C(q) n! r^{4\alpha(n,q)(p-1)} \alpha(n, q)^{-n} .$$

Then,

$$|\mathcal{E}| \leq |Z| \varphi(n-1, q) \leq kr^{3\alpha+1} C(q)(n-1)! r^{4q\alpha(n-1,q)} \alpha(n-1, q)^{1-n} .$$

Note that each $A \in \mathcal{H}(1, i_1, \ldots, i_s)$ must intersect each of r pairwise disjoint sets $X(i_1, i_2, \ldots, i_s, 1) \backslash Z$, $X(i_1, i_2, \ldots, i_s, 2) \backslash Z, \ldots, X(i_1, i_2, \ldots, i_s, r) \backslash Z$. The cardinalities of these sets for $s < m$ are at most y and for $s = m$ are less than $2y$. Consequently, by (4),

$$|\mathcal{H}(1, i_1, \ldots, i_s)| \leq (2y)^r \varphi(n-r, q) \leq \left(\frac{2n}{3\alpha}\right)^r C(q)(n-r)! r^{4q\alpha(n-r,q)} \alpha(n-r, q)^{r-n} .$$

Observe that, for $n > 20$ and $1 \leq i \leq 0.5 \log n$,

$$\frac{\alpha(n-i,q)}{\alpha(n,q)} \geq \frac{\log\log(n-i)}{\log\log n} \geq \frac{\log(\log n - i/(n-i))}{\log\log n}$$
$$\geq \frac{\log((1-1/(n+1))\log n)}{\log\log n} \geq \frac{\log\log n - 1/n}{\log\log n} \geq 1 - \frac{1}{n}.$$

Therefore,

$$|\mathcal{F}| \leq C(q)n!r^{(4p-1)\alpha(n,q)}\alpha(n,q)^{-n} + kr^{3\alpha+1}C(q)(n-1)!r^{4q\alpha(n-1,q)}$$
$$\times ((n-1)\alpha(n,q)/n)^{1-n}$$
$$+ \frac{r^{m+1}-1}{r-1}\left(\frac{2}{2.8\alpha(n,q)}\right)^{r} C(q)n!r^{4q\alpha(n-r,q)}\left(\frac{n-1}{n}\alpha(n,q)\right)^{r-n}$$
$$< C(q)n!\alpha(n,q)^{-n}r^{4p\alpha(n,q)}\left(r^{-\alpha(n,q)} + \frac{3(\log\log n)^{4q\alpha(n,q)+3}\alpha(n,q)}{\log n}\right.$$
$$\left. + \frac{3(\log\log n)^{4q\alpha(n,q)}}{1.4^{\lfloor \log\log n \rfloor}}\right).$$

But for large n the expression in big parentheses does not exceed

$$(\log n)^{-1/10q} + 3(\log\log n)^{4}(\log n)^{-4/5} + 4/2(\log n)^{0.2-\log 1.4} < 1 .$$

Thus, the theorem is proved. □

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