Amer. Math. Soc. Transl.
(2) Vol. 132, 1986

On the Minimum of the Hadwiger Number for Graphs with a Given Mean Degree of Vertices

UDC 519.1

A. V. KOSTOCHKA

Introduction

The Hadwiger number $\eta(G)$, as one of the interesting topological characteristics of graphs [1], has attracted particular attention in connection with Hadwiger's conjecture that for any graph G its Hadwiger number is not less than its chromatic number $\chi(G)$. A special case of this conjecture is the four-color problem.

Let us recall the concept of the Hadwiger number.

An elementary contraction of a graph G is one of the following operations:

a) removal of an edge;

b) removal of an isolated vertex;

c) replacement of two adjacent vertices v_1 and v_2 by a new vertex v_3 that is joined by an edge to every vertex adjacent in G to at least one of the vertices v_1 and v_2.

We say that a graph G is contracted to a graph H if H can be obtained from G by a sequence of elementary contractions. The Hadwiger number $\eta(G)$ of a graph G is the number of vertices of the maximal complete graph to which G is contracted.

A number of authors (see [5]–[7], for example) have been interested in how small a Hadwiger number a graph with a given mean degree of vertices can have. Mader [8] showed that for graphs with mean degree of vertices $2k$ the minimum Hadwiger number is greater than $k/8\log_2 k$. The aim of the present paper is to show that this minimum is a quantity of order $k/\sqrt{\ln k}$.

Let us state the problem more precisely.

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 05C35; Secondary 05C15.

Translation of Metody Diskret. Analiz. Vyp. **38** (1982), 37–58; MR **85c:05018**.

Let $\mathcal{D}_{n,k}$ (respectively, $\mathcal{E}_{n,k}$) be a set of graphs with n vertices ($n \geq k$) and with at least kn (respectively, more than $kn - \binom{k+1}{2}$) edges; and let $\mathcal{D}_k$ ($\mathcal{E}_k$) $= \bigcup_{k=n}^{\infty} \mathcal{D}_n$ ($\mathcal{E}_n$).

Our aim is to investigate $\eta(k) \doteq \min_{G \in \mathcal{D}_k} \eta(G)$ (the symbol $\doteq$ means "equals by definition").

Earlier (see [5], [6], or [8], for example), it was primarily $\eta_1(k) \doteq \min_{G \in \mathcal{E}_k} \eta(G)$ that was investigated. The result of Mader [8] mentioned above can be stated as follows: $\eta_1(k) > k/8 \log_2 k$. However, the quantities $\eta(k)$ and $\eta_1(k)$ do not differ in order. Obviously, $\eta(k) \geq \eta_1(k)$.

Let us put $w(k) = \min_{\{G/\chi(G) \geq k\}} \eta(G)$; then Hadwiger's conjecture is equivalent to the assertion that $w(k) \geq k$ for all natural numbers k. Wagner [3] showed that $w(k) \geq 4 + \log_2((k-1)/3)$. From a result of Mader [8] it follows that $w(k) > k/16 \log_2 k$.

The main results of this paper are the following.

ASSERTION 1. *For any sufficiently large k*

$$\eta_1(k) \leq 2.25k/\sqrt{\ln k}; \qquad \text{(a)}$$

$$\eta(k) \leq 3.15k/\sqrt{\ln k}. \qquad \text{(b)}$$

THEOREM. *For any $k \geq 2$*

$$\eta(k) \geq \eta_1(k) \geq 0.064k/\sqrt{\ln k} + 1.$$

COROLLARY 1. *For any $k \geq 3$*

$$w(k) \geq 0.032k/\sqrt{\ln k}.$$

PROOF. Every k-chromatic graph G contains a color-critical subgraph G' for which the degree of every vertex is at least $k-1$. Therefore, $G' \in \mathcal{D}_{[(k-1)/2]}$. (Here and later $[x]$ denotes the integral part of the real number x.) By the theorem,

$$\eta(G) \geq \eta(G') \geq 0.064(k/2-1)/\sqrt{\ln(k/2)} + 1 \geq 0.032k/\sqrt{\ln k}.$$

COROLLARY 2. *Hadwiger's conjecture is true for almost all graphs with n vertices.*

PROOF. It is known [9] that, for almost all graphs G with n vertices, $\chi(G) \leq n/\ln n$. It is also known that, for almost all graphs G with n vertices, $|E(G)| \geq n^2/6$. Hence, by the theorem, $\eta(G) \geq 0.064n/6\sqrt{\ln n}$. For large n we have $0.064n/6\sqrt{\ln n} > n/\ln n$.

REMARK. The result of Corollary 2 was obtained independently by Bollobás, Catlin, and Erdős.

COROLLARY 3. *Hadwiger's conjecture is true for almost all graphs with n vertices and kn edges if n is sufficiently large.*

PROOF. It is known ([4], Chapter 16, Exercise 9) that, if k is large, for almost all graphs G with n vertices and kn edges, $\chi(G) \leq 3k/\log_2 k$. By the theorem, for any graph G of this class, $\eta(G) \geq 0.064k/\sqrt{\ln k}$. Therefore, for large k we have $\eta(G) \geq \chi(G)$.

COROLLARY 4.

$$\min_{|V(G)|=n} (\eta(G) + \eta(\overline{G})) = O(n/\sqrt{\ln n}),$$

where $V(G)$ is the set of vertices of G, and $\overline{G}$ is the complement of G.

PROOF. Suppose that $n \geq 2$ and $|V(G)| = n$. Then the number of edges of either G or $\overline{G}$ is greater than $(n/6)n$. Therefore, by the theorem, $\max\{\eta(G), \eta(\overline{G})\} \geq 10^{-2}n/\sqrt{\ln n}$. On the other hand, putting $p = 0.5$ in the proof of Assertion 2 (in §1, below), we see that, for almost all graphs G with n vertices, $\eta(G) \leq n/\sqrt{\ln n}$. Hence there is a graph G_0 such that $\eta(G) = n/\sqrt{\ln n}$ is true for both G_0 and $\overline{G}_0$. Consequently, $\eta(G_0) + \eta(\overline{G}_0) \leq 2n/\sqrt{\ln n}$.

COROLLARY 5. *Let $\nu(k)$ be the smallest Hadwiger number that a k-connected graph can have. Then $\nu(k) = O(k/\sqrt{\ln k})$.*

Corollary 5 follows from the obvious inequality $2\nu(k) \geq \eta(k)$ and a result of Mader [8] that $\nu(k) \leq 2\eta(k)$.

Let us agree on the following notation. For any graph G let $V(G)$, $E(G)$ and $\overline{G}$ denote respectively the set of vertices, the set of edges and the complement of G. We also put $n(G) \doteq |V(G)|$ and $m(G) \doteq |E(G)|$.

If $V_0 \subset V(G)$, let $G(V_0)$ and $G \backslash V_0$ denote the subgraphs of G generated by the sets of vertices V_0 and $V(G)\backslash V_0$, respectively. If $v \in V(G)$, then $N_G(v) \doteq \{w \in V(G) | (v, w) \in E(G)\}$, $s_G(v) \doteq |N_G(v)|$, and $\sigma(G) \doteq \max_{v \in V(G)} s_G(V)$.

§1. Upper bounds for $\eta(k)$ and $\eta_1(k)$

A *pseudocoloring* of a graph G with r colors is defined as any partition of $V(G)$ into subsets $V_1, \ldots, V_r$ such that for any $1 \leq i < j \leq r$ there is an $e \in E(G)$ joining V_i and V_j. The *pseudochromatic number* $\tau(G)$ of a graph G is defined as the largest r for which there is a pseudocoloring of G with r colors.

A contraction of a graph G onto a complete graph K_r can be treated as a map $\varphi: V_0 \overset{\text{onto}}{\to} V(K_r)$, where $V_0 \subset V(G)$. If for each $v_i \in V(K_r)$ the vertices of the set $\varphi^{-1}(v_i)$ are colored with the ith color, we obtain a pseudocoloring of G with r colors. Consequently, $\tau(G) \geq \eta(G)$ for any graph G. Therefore, Assertion 1 follows from the next result.

ASSERTION 2. *If k is sufficiently large, then*
(a) $\min_{G \in \mathcal{E}(k)} \tau(G) < 2.25k/\sqrt{\ln k}$;
(b) $\min_{G \in \mathcal{D}(k)} \tau(G) < 3.15k/\sqrt{\ln k}$.

PROOF. Suppose that $0.5 \leq p < 1$ and that $G_{n,p}$ is a random variable whose admissible values are elements of the set of all n-vertex graphs with numbered vertices, and $P(e \in G_{n,p}) = p$ for every edge e and for different edges these probabilities are independent.

Let us compute the probability $\Psi(k_1, \dots, k_r)$ of the appearance of n-vertex graphs for which a fixed partition $\mathcal{M}$ of the set $V(G)$ into r subsets $M_1, \dots, M_r$ with $|M_i| = k_i$ $(i = 1, \dots, r)$ is a pseudocoloring. The probability that there is an edge between M_i and M_j is $1 - (1-p)^{k_i k_j}$. Hence,

$$\Psi(k_1, \dots, k_r) = \prod_{i=1}^{r-1} \prod_{j=i+1}^{r} (1 - (1-p)^{k_i k_j}).$$

By any method for finding a numerical extremum it is easy to establish that

$$\max_{\substack{k_1+\dots+k_r=n \\ k_1,\dots,k_r \geq 1}} \Psi(k_1, k_2, \dots, k_r) = \Psi\left(\frac{n}{r}, \frac{n}{r}, \dots, \frac{n}{r}\right) = (1 - (1-p)^{n^2/r^2})^{r(r-1)/2}.$$

Since the number of partitions of an n-element set into r subsets is less than n^r, the probability of the appearance of an n-vertex graph G with $\tau(G) \geq r$ does not exceed

$$n^n (1 - (1-p)^{n^2/r^2})^{r(r-1)/2} < \exp\{n \cdot \ln n - (1-p)^{n^2/r^2} r^2/3\}. \qquad (*)$$

If for some $\delta > 1/(1-p)$ we have $r \geq n/\sqrt{\log_\delta n}$ then the right-hand side of $(*)$ does not exceed

$$\exp\left\{n \ln n - \frac{n^2(1-p)^{\log_\delta n}}{3 \log_\delta n}\right\} = \exp\left\{n\left(\ln n - \frac{n^{1+\log_\delta(1-p)}}{3\log_\delta n}\right)\right\} \underset{n\to\infty}{\longrightarrow} 0.$$

Suppose that $p = 2/3$ and $r \geq n/\sqrt{\log_{3.01} n}$. Then for large n the probability that $\tau(G_{n,p}) \geq r$ is less than 10^{-3}. On the other hand, by the de Moivre-Laplace theorem, for large n the probability that $m(G_{n,p}) \geq n^2/3$ is greater than 0.01. Hence there is a graph G_n such that $n(G_n) = n$, $m(G_n) \geq n^2/3$, and $\tau(G_n) < n/\sqrt{\log_{3.01} n}$. If n is divisible by 3 and $k = n/3$, then $G_n \in \mathcal{D}_k$ and

$$\tau(G_n) \leq \frac{3k}{\sqrt{\log_{3.01} 3k}} < \frac{3k\sqrt{\ln 3.01}}{\sqrt{\ln k}} \leq 3.15 \cdot \frac{k}{\sqrt{\ln k}}.$$

This proves (b).

Similarly, suppose that $p = 8/9$ and $r \geq n/\sqrt{\log_{9.2} n}$. Then, as above, for sufficiently large n there is a graph H_n such that $n(H_n) = n$, $m(H_n) \geq \frac{4}{9}n^2$, and $\tau(H_n) < n/\sqrt{\log_{9.2} n}$. If n is divisible by 3 and $k = 2n/3$, then $H_n \in \mathcal{E}_k$ and

$$\tau(H_n) \leq \frac{3k/2}{\sqrt{\log_{9.2}(3k/2)}} \leq \frac{3k\sqrt{\ln 9.2}}{2\sqrt{\ln k}} \leq 2.25 \frac{k}{\sqrt{\ln k}}.$$

This proves the assertion.

§2. Idea of the proof of the theorem

The main part of the proof consists in studying contractions of graphs in which the number of edges is substantially greater than in their complements.

The main role is played by a function of real variables

$$f(n;m;\beta) \doteq \frac{n}{2\sqrt{\ln n}}\left(\sqrt{\ln\frac{n^2}{2m}}-\beta\right).$$

For any graph G we put $f(G;\beta) \doteq f(n(G);m(G);\beta)$. By means of $f(G;\beta)$ we shall give lower bounds for $\eta(\overline{G})$. In considering differences of the form $f(G;\beta) - f(H;\beta)$ we need the function

$$\varphi(c;z;\beta) \doteq (c-z)\cdot\sqrt{\ln\frac{c-z}{1+\ln c-\beta\sqrt{\ln c}-z}} - c\sqrt{\ln c}+\beta z.$$

In Lemmas 1–6 we determine analytic properties of the functions f and φ that we need in the proof of the later lemmas. Since the proofs of Lemmas 1–6 do not relate to graph theory, they are given in the Appendix.

The key result is Lemma 7. In it we prove that any n-vertex graph G with comparatively large minimal degree can be contracted to an $\lceil n/2\rceil$-vertex graph G' such that the number of edges in $\overline{G}'$ is substantially less than in $\overline{G}$. On the basis of Lemma 7 we show in Lemmas 8 and 9 that, for some β and restrictions on the number of edges in $\overline{G}$, the graph G can be contracted to a graph H with fewer edges such that $f(\overline{H};\beta) \geq f(\overline{G};\beta)$. Using this, in Lemma 10 we prove that, for graphs G whose complement contains less than 1/13 of the number of all edges,

$$\eta(G) \geq f(\overline{G};1.1).$$

Moreover, in Lemmas 11–15 we successively derive bounds analogous to those in Lemma 10 under weaker restrictions on the number of edges in $\overline{G}$. We use lower bounds for certain values of the function $\varphi(c;z;\beta)$, given in the table at the end of the paper.

The proof of the theorem really consists in reducing the general case to the case considered in Lemma 15.

LEMMA 1. *For any real $x \geq 2$ and $\beta \geq 3/5$*

$$x-1-2\ln x+2\beta\sqrt{\ln x} > 0.$$

LEMMA 2. *If $n \geq 4$, $2 \leq 2m \leq n$, and $0.6 \leq \beta < \sqrt{\ln n^2/2m}$, then $f(n-1;m;\beta)+1 \geq f(n;m;\beta)$, with f as defined at the beginning of* §2.

LEMMA 3. *Suppose that $2m \geq n \geq 4$ and*

$$0.6 \leq \beta < \sqrt{\ln\frac{n^2}{2m}}, \qquad \frac{n}{2} > \alpha \geq 1+\ln\frac{n^2}{2m}-\beta\sqrt{\ln\frac{n^2}{2m}}.$$

Then

$$f(n;m;\beta) \leq f(n-1;m-2m\alpha/n;\beta).$$

LEMMA 4. *Suppose that $c \geq 2$, $\beta \geq 0.6$, and $1 \leq z < 1+\ln c-\beta\sqrt{\ln c}$. Then $\varphi'_z(c;z;\beta) > 0$, where φ was defined at the beginning of* §2.

LEMMA 5. *Suppose* $a \geq 2$, $\beta \geq 0.6$, $1 \leq z < 1 + \ln a - \beta\sqrt{\ln a}$, *and* $c \geq a$. *Then* $\varphi_c'(c; z; \beta) < 0$.

LEMMA 6. *Suppose that* $g(x) = e^{1.1x}/4 + 1.1x - x^2 - e^{1.1x - x^2}/4$. *Then* $g(x) > 0$ *for any* $x \geq 1.7$.

§3. Lemma on contractions

LEMMA 7. *Suppose that* $k \geq 6$ *and that* G *is an* n*-vertex graph such that* $\sigma(G) \leq n/k$. *Then* $\overline{G}$ *can be contracted to an* $[n/2]$*-vertex graph* $\overline{H}$ *such that*

$$2m(H) \leq \frac{27k}{(k-2)^3(k-1)^2} n^2(H).$$

PROOF. We join each vertex $v \in V(G)$ by quasi-edges to those vertices not adjacent to v that are joined to v by more than $3n/(k-1)(k-2)$ paths of length 2. Then any vertex v is incident with no more than $n(k-2)/3k - 1$ quasi-edges. For otherwise the total number of paths of length 2 with v at one end would be greater than

$$\frac{3n}{(k-1)(k-2)} \cdot \frac{n(k-2) - 3k}{3k}.$$

However, the number of such paths does not exceed $\sigma(G)(\sigma(G) - 1) \leq n(n-k)/k^2$, and the inequality

$$\frac{3n}{(k-1)(k-2)} \cdot \frac{n(k-2) - 3k}{3k} > \frac{n(n-k)}{k^2}$$

is true for any $n \geq k \geq 6$.

Let G' denote the graph obtained from G by adding quasi-edges. Then

$$\sigma(G') \leq \frac{n}{k} + \frac{n(k-2)}{3k} - 1 = \frac{n(k+1)}{3k} - 1 < \frac{n-1}{2}.$$

Hence there is a Hamiltonian cycle in $\overline{G}'$ and there is a pairing of $[n/2]$ edges; that is, in G' there is a system π of $[n/2]$ pairs of vertices such that the vertices of any pair are not joined by an edge.

To a contraction of an edge (v_1, v_2) in $\overline{G}$ there corresponds in G an identification; that is, the replacement of the nonadjacent vertices v_1 and v_2 by a new vertex $\tilde{v}$ that is adjacent to just those vertices that are adjacent in G to both v_1 and v_2.

Suppose we are given an arbitrary system π of $[n/2]$ pairs of vertices of G, corresponding to a pairing in $\overline{G}'$. Identifying the vertices of each pair in G and rejecting the unpaired vertex if n is odd, we obtain a graph H_π. Among the H_π we choose the $H \doteq H_{\pi_0}$ with the fewest edges. Let $V(H) = \{v_i | i = 1, \dots, [n/2]\}$ and suppose that v_i is obtained by identifying v_{1i} and v_{2i} from $V(G)$. We observe that

$$(v_i, v_j) \in E(H) \Leftrightarrow \{(v_{1i}, v_{1j}), (v_{1i}, v_{2j}), (v_{2i}, v_{1j}), (v_{2i}, v_{2j})\} \subset E(G).$$

We define a *fork* on a pair of vertices x_1 and x_2 of $V(G)$ as a path (x_1, x_3, x_2) of length 2 joining x_1 and x_2 in G. We shall call (x_1, x_2) the *support* of this fork.

If (v_{1i}, v_{1j}), $(v_{2i}, v_{2j}) \notin E(G)$ (respectively (v_{1i}, v_{2j}), $(v_{2i}, v_{1j}) \notin E(G)$) for some $i \neq j$, i, $j \in \{1, 2, \dots, [n/2]\}$, then the $(i,j,1)$-transform (respectively, $(i,j,2)$-transform) of H is the graph H^1_{ij} (respectively, H^2_{ij}) corresponding to the system π^1_{ij} (π^2_{ij}) that differs from π_0 only in the fact that the pairs $\{v_{1i}, v_{2i}\}$ and $\{v_{1j}, v_{2j}\}$ are replaced by the pairs $\{v_{1i}, v_{1j}\}$ and $\{v_{2i}, v_{2j}\}$ (respectively, by $\{v_{1i}, v_{2j}\}$ and $\{v_{2i}, v_{1j}\}$). Each of these transforms will be called a (i,j)-*transform*.

The edges of $E(G)$ hinder the existence of certain (i,j)-transforms. Since any edge can hinder no more than one transform, every vertex $v_i \in V(H)$ can occur in at least $n-3-2(n(k+1)/3k-1)$ transforms. Let $s_H(v_i) = r_i$ $(1 \le i \le [n/2])$. In the (i,j,l)-transform, $r_i + r_j$ edges of $E(H) \backslash E(H^l_{ij})$ are destroyed, but new edges may be formed. By virtue of the choice of H, for any $1 \le i$, $j \le [n/2]$ and $1 \le l \le 2$ we necessarily have $|E(H^l_{ij}) \backslash E(H)| \ge r_i + r_j$.

To each edge of $E(H^l_{ij}) \backslash E(H)$ there corresponds in G a pair of forks that has a common support from $\pi_0 \cap \pi^l_{ij}$. Any pair of forks can occur in only one transform in the creation of a "new" edge. Consequently, to each (i,j,l)-transform there correspond $x^l_{ij} \ge r_i + r_j$ pairs of forks, and any pair of such forks has a common support belonging to π_0. We associate any r_i of these pairs of forks with the vertex v_i, and any r_j of the remaining $x^l_{ij} - r_i$ pairs of forks with the vertex v_j. In all, to a vertex v_i $(1 \le i \le [n/2])$ there correspond no fewer than $r_i(n - (2n(k+1)/3k) - 1)$ pairs of forks. Moreover, to each edge $(v_i, v_j) \in E(H)$ there correspond the pairs of forks

$$\{(v_{1i}, v_{1j}, v_{2i}),\ (v_{1i}, v_{2j}, v_{2i})\}$$

and

$$\{(v_{1j}, v_{1i}, v_{2j}),\ (v_{1j}, v_{2i}, v_{2j})\},$$

which do not correspond to one vertex. Hence the total number of pairs of forks in G on pairs of vertices of π_0 is not less than

$$\sum_{i=1}^{[n/2]} r_i \left(\frac{n(k-2)}{3k} - 1\right) + 2m(H) = \left(\frac{n(k-2)}{3k} - 1\right) \cdot 2m(H) + 2m(H)$$
$$= 2m(H)\frac{n(k-2)}{3k}.$$

On the other hand, since $(v_{1j}, v_{2j}) \notin E(G')$ for any $1 \le j \le [n/2]$, there are no more than $3n/(k-1)(k-2)$ forks on $\{v_{1j}, v_{2j}\}$ and consequently no more than $\binom{3n/(k-1)(k-2)}{2}$ pairs of forks. Hence,

$$\frac{2m(H) \cdot n(k-2)}{3k} \le \left[\frac{n}{2}\right] \cdot \frac{9n}{2(k-1)^2(k-2)^2} \cdot \left(n - \frac{(k-1)(k-2)}{3}\right),$$

and

$$2m(H) \le \left[\frac{n}{2}\right] \cdot \frac{n-1}{2} \cdot \frac{27k}{(k-1)^2(k-2)^3}.$$

This proves the lemma.

LEMMA 8. *Let G be an n-vertex graph. Put $c = n^2/2m(G)$ and $z = \sigma(G)c/n$. Suppose that $c \geq 8.5z$ and $z \leq e^{\sqrt{\ln c}}/3e^{0.25}$. Then $\overline{G}$ can be contracted to an $[n/2]$-vertex graph $\overline{H}$ such that either $f(H;1) \geq f(G;1)$ or $\overline{H} = K_{[n/2]}$.*

PROOF. By Lemma 7, $\overline{G}$ can be contracted to an $[n/2]$-vertex graph $\overline{H}$ such that $E(H) = \varnothing$ or

$$c_1 \doteq \frac{n^2(H)}{2m(H)} \geq \left(\frac{c}{z}\right)^4 \cdot \frac{1}{27} \cdot \left(1 - \frac{z}{c}\right)^2 \cdot \left(1 - \frac{27}{c}\right)^3.$$

Suppose that $E(H) \neq \varnothing$. For $z/c \leq 1/8.5$ we have $(1 - z/c)^2(1 - 27/c)^3 > \frac{1}{3}$. Hence, $c_1 \geq (c/z)^4/81$ and

$$\begin{aligned} f(H;1) - f(G;1) &= \frac{[n/2](\sqrt{\ln c_1} - 1)}{2\sqrt{\ln[n/2]}} - \frac{n(\sqrt{\ln c} - 1)}{2\sqrt{\ln n}} \\ &\geq \frac{n}{2\sqrt{\ln n}}\left(\frac{1}{2}\sqrt{\ln c_1} - \sqrt{\ln c} + \frac{1}{2}\right) \geq \frac{n}{2\sqrt{\ln n}}\left(\sqrt{\ln(c/3z)} - \sqrt{\ln c} + \frac{1}{2}\right) \\ &= \frac{n}{2\sqrt{\ln n}} \cdot \frac{\ln c - \ln 3z - \ln c + \sqrt{\ln c} - 0.25}{\sqrt{\ln(c/3z)} + \sqrt{\ln c} - 1/2}. \end{aligned}$$

Thus, it is sufficient to prove the inequality $\ln 3z \leq \sqrt{\ln c} - 0.25$, which is equivalent to the condition $e^{\sqrt{\ln c}}/3e^{0.25} \geq z$ of our lemma.

LEMMA 9. *Let G be an n-vertex graph. Put $c = n^2/2m(G)$ and $z = \sigma(G)c/n$. Suppose that $c \geq 13$ and $z \leq e^{1.1\sqrt{\ln c}}/4$. Then $\overline{G}$ can be contracted to an $[n/2]$-vertex graph $\overline{H}$ such that either $f(H;1.1) \geq f(G;1.1)$ or $\overline{H} = K_{[n/2]}$.*

PROOF. By Lemma 7, $\overline{G}$ can be contracted to an $[n/2]$-vertex graph $\overline{H}$ such that either $E(H) = \varnothing$ or

$$c_1 \doteq \frac{n^2(H)}{2m(H)} \geq \left(\frac{c}{z}\right)^4 \cdot \frac{1}{27} \cdot \left(1 - \frac{z}{c}\right)^2 \cdot \left(1 - \frac{27}{c}\right).$$

Suppose that $E(H) \neq \varnothing$. For $c \geq 13$ we have $e^{1.1\sqrt{\ln c}}/4c \leq 1/8.9$, and for $c/z \geq 8.9$ we have

$$\left(1 - \frac{z}{c}\right)^2 \left(1 - \frac{27}{c}\right)^3 \geq \frac{27}{76}.$$

Consequently, $c_1 \geq (c/z)^4/76$ and

$$\begin{aligned} f(H;1.1) - f(G;1.1) &= \frac{[n/2](\sqrt{\ln c_1} - 1.1)}{2\sqrt{\ln[n/2]}} - \frac{n(\sqrt{\ln c} - 1.1)}{2\sqrt{\ln n}} \\ &\geq \frac{n}{2\sqrt{\ln n}}\left(\frac{1}{2}\sqrt{\ln c_1} - \sqrt{\ln c} + 0.55\right) \\ &= \frac{n}{2\sqrt{\ln n}} \cdot \frac{0.25\ln c_1 - (\sqrt{\ln c} - 0.55)^2}{0.5\sqrt{\ln c_1} + \sqrt{\ln c} - 0.55} \\ &\geq \frac{n}{2\sqrt{\ln n}} \cdot \frac{\ln c - \ln z - 0.25\ln 76 - \ln c + 1.1\sqrt{\ln c} - 0.3025}{0.5\sqrt{\ln c_1} + \sqrt{\ln c} - 0.55}. \end{aligned}$$

Now to prove the inequality $f(H;1.1) \geq f(G;1.1)$ it is sufficient to check that $\ln z \leq 1.1\sqrt{\ln c} - 0.25(1.21 + \ln 76)$. But since $\exp\left\{\frac{1}{4}(1.21 + \ln 76)\right\} < 4$, this inequality follows from the condition $z \leq e^{1.1\sqrt{\ln c}}/4$ of our lemma.

§4. Lower bounds for $\eta(\overline{G})$ expressed in terms of $f(G;\beta)$

LEMMA 10. *Let G be an n-vertex graph, and suppose that* $0 < 2m(G) \leq n^2/13$. *Then* $\eta(\overline{G}) \geq f(G;1.1)$.

PROOF. It is easy to verify the lemma for $n \leq 13$. Let G be a graph with the smallest number of vertices for which the lemma is false. Then $n \geq 14$. We consider five cases.

Case 1. Suppose that $0 < 2m(G) < n$. Then in G there is an isolated vertex v_1. Let $H = G\backslash\{v_1\}$. By the inductive hypothesis (IH), $\eta(\overline{H}) \geq f(H;1.1)$. But $\eta(\overline{G}) = \eta(\overline{H}) + 1 \geq f(H;1.1) + 1$, and by Lemma 2 we have $f(H;1.1) + 1 \geq f(G;1.1)$. For the rest of the lemma we assume that $c = n^2/2m(G)$ and $\sigma(G) = s_G(v_0) = zn/c$.

Case 2. Suppose that $2m(G) \geq n$ and $z \leq e^{1.1\sqrt{\ln c}}/4$. Since $c \leq n$ we have $f(G;1.1) < [n/2]$. Let H be the graph with $[n/2]$ vertices from the statement of Lemma 9. If $E(H) = \varnothing$, then $\eta(\overline{G}) \geq [n/2] \geq f(G;1.1)$. If $m(H) > 0$, then, by Lemma 7, $n^2(H)/2m(H) > 13$, and by Lemma 9, $f(H;1.1) \geq f(G;1.1)$. Consequently, by the IH, $\eta(\overline{H}) \geq f(H;1.1) \geq f(G;1.1)$.

Case 3. Suppose that $2m(G) \geq n$ and

$$s_G(v_0) + \sigma(G\backslash N(v_0)) \geq n \cdot (1 + \ln c - 1.1\sqrt{\ln c})/c.$$

Suppose also that $v_1 \in V(G)\backslash N(v_0)$ and $s_{G\backslash N(v_0)}(v_1) = \sigma(G\backslash N(v_0))$. Then identifying v_0 and v_1 we obtain a graph G' with $n(G') = n - 1$ and

$$m(G') = m(G) - s_G(v_0) - s_{G\backslash N(v_0)}(v_1) \leq m(G) - 2m(G)/n(1 + \ln c - 1.1\sqrt{\ln c}).$$

It is easy to check that for $c \geq 13$ we have $2m(G') \leq (n-1)^2/13$, and, by the IH and Lemma 3,

$$\eta(\overline{G}) \geq \eta(\overline{G}') \geq f(G';1.1) \geq f(G;1.1).$$

Case 4. The conditions of Cases 1–3 are not satisfied and $c \geq e^{2.89}$. Consider $G' \doteq G\backslash N(v)$. Then $n(G') = ((c-z)/c) \cdot n$ and

$$\sigma(G') < (n/c)(1 + \ln c - 1.1\sqrt{\ln c} - e^{1.1\sqrt{\ln c}}/4).$$

Let us put $k = n(G')/\sigma(G')$. Putting $x = \sqrt{\ln c}$ in Lemma 6, we have $k \geq c$. Let us apply Lemma 7 to G'. If $n(G')$ is odd, we take v_0 for a vertex that does not occur in one of the identified pairs. Taking account of the fact that v_0 is an isolated vertex in G', we find that $\overline{G}$ can be contracted to an $[(n(G')+1)/2]$-vertex graph $\overline{H}$ such that

$$\begin{aligned} 2m(H) &\leq \frac{n^2(H) \cdot 27k}{(k-1)^2(k-2)^3} \leq \frac{n^2(H) \cdot 27c}{(c-1)^2(c-2)^3} \\ &\leq \frac{n^2(H) \cdot 27}{c^4}\left(1 - \frac{1}{17.99}\right)^{-2}\left(1 - \frac{2}{17.99}\right)^{-3} \leq \frac{e^4}{c^4} \cdot n^2(H). \end{aligned}$$

If $m(H) = 0$, then

$$\eta(\overline{G}) \geq \eta(\overline{G}') \geq n(H) \geq \tfrac{1}{2}n(1 - z/c) \geq \tfrac{1}{2}n(1 - \ln c/c).$$

Since $e^{2.89} \leq c \leq n$, we have

$$\frac{n}{2}\left(1 - \frac{\ln c}{c}\right) \geq \frac{n}{2}\left(1 - \frac{1}{\sqrt{\ln c}}\right) \geq \frac{n(\sqrt{\ln c} - 1)}{2\sqrt{\ln n}} \geq f(G; 1.1).$$

Suppose that $m(H) > 0$. We show that $f(H; 1.1) \geq f(G; 1.1)$. In fact,

$$\begin{aligned} f(H;1.1) - f(G;1.1) &\geq \frac{n(1 - z/c)(\sqrt{\ln(c^4/e^4)} - 1.1)}{2 \cdot 2\sqrt{\ln(n(1 - z/c)/2)}} - \frac{n(\sqrt{\ln c} - 1.1)}{2\sqrt{\ln n}} \\ &\geq \frac{n}{2c\sqrt{\ln n}}((c - z)(\sqrt{\ln c - 1} - 0.55) - c(\sqrt{\ln c} - 1.1)) \\ &\geq \frac{n}{2c\sqrt{\ln n}}((c - \ln c)\sqrt{\ln c - 1} + 0.55c + 0.55\ln c - c\sqrt{\ln c}) \\ &\geq \frac{n}{2c\sqrt{\ln n}}\left(c \cdot \frac{-1}{\sqrt{\ln c - 1} + \sqrt{\ln c}} + 0.55c - \ln c(\sqrt{\ln c - 1} - 0.55)\right). \end{aligned}$$

Since $c \geq e^{2.89}$, we have

$$\begin{aligned} f(H;1.1) - f(G;1.1) &\geq \frac{n}{2c\sqrt{\ln n}}\left(\frac{-c}{1.37 + 1.7} + 0.55c - \ln^{3/2} c + 0.55 \cdot 2.89\right) \\ &\geq \frac{n}{2c\sqrt{\ln n}}(0.224c + 1.58 - \ln^{3/2} c) > 0. \end{aligned}$$

From the form of the function f and the fact that $f(H; 1.1) \geq f(G; 1.1)$ it follows that $c_1 \doteq n^2(H)/2m(H) > c$. Then, by the IH, $\eta(\overline{H}) \geq f(H; 1.1) \geq f(G; 1.1)$.

Case 5. The conditions of Cases 1–3 are not satisfied and $c \in [13, e^{2.89}]$. Let us consider $G' \doteq G \backslash N(v_0)$. As in Case 4, $n(G') = ((c - z)/c)n$, and

$$m(G') \leq \frac{n(G') \cdot \sigma(G')}{2} \leq \frac{(c - z) \cdot n^2}{2c^2}(1 + \ln c - 1.1\sqrt{\ln c} - z).$$

If $m(G') = 0$, then $\eta(\overline{G}) \geq n(c - z)/c \geq f(G; 1.1)$. Suppose that $m(G') > 0$. Then

$$c_1 = n^2(G')/2m(G) \geq (c - z)(1 + \ln c - 1.1\sqrt{\ln c} - z)$$

We show that $f(G'; 1.1) \geq f(G; 1.1)$. In fact,

$$\begin{aligned} f(G';1.1) - f(G;1.1) &\geq \frac{c - z}{c} \cdot \frac{n\left(\sqrt{\ln \frac{c - z}{1 + \ln c - 1.1\sqrt{\ln c} - z}} - 1.1\right)}{2\sqrt{\ln n}} \\ &\quad - \frac{n(\sqrt{\ln c} - 1.1)}{2\sqrt{\ln n}} \\ &= \frac{n}{2c\sqrt{\ln n}} \cdot \varphi(c; z; \beta), \end{aligned}$$

where $\varphi(c; z; \beta)$ was defined at the beginning of §2.

By Lemmas 4 and 5, to check that $\varphi(c;z;\beta) \geq 0$ for $c \in [13, e^{2.89}]$ and $z \geq e^{1.1\sqrt{\ln c}}/4$ it is sufficient to verify it for $c = e^{2.89}$ and any $z \leq e^{1.1\sqrt{\ln 13}}/4$. In the table at the end of the paper it is verified for $z = 1.45$.

Thus, $f(G';1.1) \geq f(G;1.1)$ and $n(G') < n$. From the definition of f we have $n^2(G')/2m(G') > c \geq 13$, and by the IH we have $\eta(\overline{G}') \geq f(G';1.1) \geq f(G;1.1)$. This proves the lemma.

LEMMA 11. *Let G be an n-vertex graph and suppose that $0 < 2m(G) \leq n^2/6.2$. Then $\eta(\overline{G}) \geq 5f(G;1)/6$.*

PROOF. For $n \leq 7$ the lemma is obvious. Let G be the smallest counterexample. If $0 < 2m(G) < n$, then there is an isolated vertex v_1 in G. Since $2m(G) \leq n-1$, and $n \geq 8$, by the IH we have

$$\eta(\overline{G}\backslash\{v_1\}) \geq 5f(G\backslash\{v_1\};1)/6.$$

But $\eta(\overline{G}) = \eta(\overline{G}\backslash\{v_1\}) + 1$, and $5f(G\backslash\{v_1\};1)/6 + 1 \geq 5f(G)/6$ by Lemma 2. Thus, $2m(G) \geq n$.

If $2m(G) \leq n^2/13$, then $5f(G;1)/6 \leq f(G;1.1)$, and taking account of Lemma 10 we have $\eta(\overline{G}) \geq 5f(G;1)/6$.

We may thus assume that $c \in [6.2, 13]$, where $c = n^2/2m(G)$ and $\sigma(G) = s_G(v_0) = zn/c$. By Lemma 8, for $c \geq 9.4$ we necessarily have $z \geq e^{\sqrt{\ln c}}/3e^{0.25}$. Otherwise $\overline{G}$ could be contracted to a graph H with $f(H;1) \geq f(G;1)$. Let us put $G' = G\backslash N(v_0)$. Then $n(G') = (c-z)n/c$. As in Case 3 of Lemma 10, it is easy to check that it suffices to consider the case

$$zn/c + \sigma(G') < (1 + \ln c - \sqrt{\ln c})n/c.$$

Then

$$m(G') \leq \frac{n(G')\cdot\sigma(G')}{2} \leq \frac{n^2(c-z)(1+\ln c - \sqrt{\ln c} - z)}{2c^2}.$$

If $m(G') = 0$, then $\eta(\overline{G}) \geq (c-z)n/c > 5f(G;1)/6$, since $z < \ln c$. Suppose that $m(G') > 0$. Then

$$\begin{aligned} &\frac{5}{6}f(G';1) - \frac{5}{6}f(G;1) \\ &\geq \frac{5}{6}\left(\frac{n(c-z)\left(\sqrt{\ln \frac{c-z}{1+\ln c\sqrt{\ln c} - z}} - 1\right)}{2c\sqrt{\ln \frac{n(c-z)}{c}}} - \frac{n(\sqrt{\ln c} - 1)}{2\sqrt{\ln n}}\right) \\ &\geq \frac{5n}{12c\sqrt{\ln n}}\cdot\varphi(c;z;1). \end{aligned}$$

By Lemmas 4 and 5, to prove that $\varphi(c;z;1) \geq 0$ for $c \in [a,b]$ and for $e^{\sqrt{\ln c}}/3e^{0.25} \leq z < 1 + \ln c - \sqrt{\ln c}$ it is sufficient to verify it for $c = b$ and any $z \leq \exp\{\sqrt{\ln a} - 0.25\}/3$. Hence, the corresponding calculations in the table confirm the truth of this inequality on the intervals [9.4; 10.1], [10.1; 10.67], [10.67; 11.2], and [12; 13]. Similarly we consider $c \in [6.2; 9.4]$. We merely need to take account of the fact that, by definition, $z \geq 1$. Therefore, $f(G'; 1) \geq f(G; 1)$ and by the IH we have $\eta(\overline{G}') \geq 5f(G'; 1)/6$. Hence, $\eta(\overline{G}) \geq \eta(\overline{G}') \geq 5f(G; 1)/6$.

As in Lemma 11, by using the table we can prove the following facts.

LEMMA 12. *Let G be an n-vertex graph, and suppose that* $0 < 2m(G) \leq n^2/4.5$. *Then* $\eta(\overline{G}) \geq (35/57)f(G; 7/8)$.

LEMMA 13. *Let G be an n-vertex graph, and suppose that* $0 < 2m(G) \leq n^2/3.15$. *Then* $\eta(\overline{G}) \geq 0.4616f(G; 0.75)$.

LEMMA 14. *Let G be an n-vertex graph, and suppose that* $0 < 2m(G) \leq n^2/2.55$. *Then* $\eta(\overline{G}) \geq 0.3663f(G; 2/3)$.

LEMMA 15. *Let G be an n-vertex graph, and suppose that* $0 < 2m(G) \leq n^2/2$. *Then* $\eta(\overline{G}) \geq 0.299f(G; 0.6)$.

§5. Proof of the theorem

From the results of Mader [8] and Tashkinov (course paper, Novosibirsk State University, 1976) for $k \leq 5$, it follows that the theorem is true for $k \leq 100$. Thus, suppose that $k > 100$ and that $G \in \mathcal{E}_k$ is a counterexample to the theorem with the smallest number of edges for the given k. If $n(G) = k$, then $m(G) > k^2 - \binom{k+1}{2} = \binom{k}{2}$, which is impossible. Hence, $n(G) > k$. By the choice of G,

$$m(G) = k \cdot n(G) - \binom{k+1}{2} + 1.$$

Under a contraction of the edge $(u, v) \in E(G)$ the number of edges in the resulting graph G' is less than $|E(G) \setminus \{(u, v)\}|$ by the number of vertices in G adjacent to both u and v, that is, by the number of triangles containing the edge (u, v). Hence, if (u, v) occurs in fewer than k triangles, we have

$$m(G') \geq m(G) - (k-1) - 1 = k \cdot n(G) - \binom{k+1}{2} + 1 - k = k \cdot n(G') - \binom{k+1}{2} + 1.$$

By the IH, we have $\eta(G') \geq 0.064k/\sqrt{\ln k} + 1$, and $\eta(G) \geq \eta(G')$. Consequently, every edge of $E(G)$ occurs in no fewer than k triangles. Thus, for any vertex v the degree of each vertex $w \in N_G(v)$ in the graph $G(N(v))$ is not less than k. Suppose that $\min_{v \in V(G)} s_G(v) = s_G(v_0) = \alpha k$. By what we stated above, $\alpha > 1$. Since $m(G)/n(G) < k$, we have $\alpha < 2$. Let us put $G_0 = G(N(v_0))$. Then $\sigma(\overline{G}_0) \leq \alpha k - 1 - k < \alpha k/2$. Consequently, by Lemma 15,

$$\eta(G_0) = 0.299 \frac{\sqrt{\ln(\alpha/(\alpha - 1))} - 0.6}{2\sqrt{\ln \alpha \cdot k}} \cdot \alpha \cdot k.$$

Let $\Psi(\alpha) \doteq \alpha(\sqrt{\ln(\alpha/(\alpha-1))} - 0.6)$; then

$$\Psi'(\alpha) = \frac{2\ln(\alpha/(\alpha-1)) - 1/(\alpha-1) - 1.2\sqrt{\ln(\alpha/(\alpha-1))}}{2\sqrt{\ln(\alpha/(\alpha-1))}}.$$

Putting $\gamma \doteq \alpha/(\alpha-1)$, we have $\gamma > 2$ and

$$\Psi'(\alpha) = (2\ln\gamma - \gamma + 1 - 1.2\sqrt{\ln\gamma})/2\sqrt{\ln\gamma}.$$

By Lemma 1, $\Psi'(\alpha) \leq 0$. Hence, taking account of the fact that $1/\sqrt{k\ln\alpha}$ decreases as α increases, we have

$$\begin{aligned}\eta(g_0) &\geq 0.299 \cdot \frac{\sqrt{\ln(2/(2-1))} - 0.6}{2\sqrt{\ln 2k}} \cdot 2k \\ &\geq 0.299 \cdot \frac{0.832 - 0.6}{\sqrt{\ln 2k}} \cdot k \geq 0.0692 \frac{k}{\sqrt{\ln 2k}} \\ &\geq \frac{k}{\sqrt{\ln k}} \cdot \frac{0.0692}{\sqrt{1 + 1/\log_2 100}} \geq 0.064 \cdot \frac{k}{\sqrt{\ln k}}.\end{aligned}$$

but since v_0 is adjacent to all the vertices of $V(G_0)$, we have $\eta(G) \geq \eta(G_0) + 1$. This proves the theorem.

BIBLIOGRAPHY

1. K. Wagner, *Über eine Eigenschaft der ebenen Komplexe*, Math. Ann. **114** (1937), 570–590.

2. H. Hadwiger, *Über eine Klassifikation der Streckenkomplexe*, Vierteljahrsschrift Naturforsch. Ges. Zürich **88** (1943), 133–142.

3. K. Wagner, *Beweis einer Abschwächung der Hadwiger-Vermutung*, Math. Ann. **153** (1964), 139–141.

4. Paul Erdős and Joel Spencer, *Probabilistic methods in combinatorics*, Academic Press, 1974.

5. A. A. Zykov, *On the number of edges of a graph with Hadwiger number not greater than* 4, Prikl. Mat. i Programmirovanie Vyp. 7 (1972), 52–55. (Russian)

6. Bohdan Zelinka, *On some graph-theoretical problems of V. G. Vising*, Časopio Pěst. Mat. **98** (1973), 56–66.

7. Zevi Miller, *Contractions of graphs: a theorem of Ore and an extremal problem*, Discrete Math. **21** (1978), 261–272.

8. W. Mader, *Homomorphiesätze für Graphen*, Math. Ann. **178** (1968), 154–168.

9. A. D. Korshunov, *On the chromatic number of n-vertex graphs*, Metody Diskret. Anal. Vyp. **35** (1980), 15–44. (Russian)

Appendix

PROOF OF LEMMA 1. The assertion of the lemma is true for $x = 2$, and for $x \geq 2$ we have

$$(x - 1 - 2\ln x + 2\beta\sqrt{\ln x})'_x = 1 - \frac{2}{x} + \frac{\beta}{x \cdot \sqrt{\ln x}} > 0.$$

PROOF OF LEMMA 2. Let us put $c = n^2/2m$. We note that, by hypothesis, $c \geq n$. Then

$$
\begin{aligned}
&f(n-1;m;\beta) - f(n;m;\beta) + 1 \\
&\quad = \frac{n-1}{2\sqrt{\ln(n-1)}}\left(\sqrt{\ln\frac{(n-1)^2}{2m}} - \beta\right) - \frac{n(\sqrt{\ln c} - \beta)}{2\sqrt{\ln n}} + 1 \\
&\quad \geq \frac{n}{2\sqrt{\ln n}}\left[\left(1 - \frac{1}{n}\right)(\sqrt{\ln c - 2/n} - \beta) - \sqrt{\ln c} + \beta + \frac{2}{n}\sqrt{\ln n}\right] \\
&\quad \geq \frac{n}{2\sqrt{\ln n}}\left[\left(1 - \frac{1}{n}\right)\left(\sqrt{\ln c} - \frac{1}{n} - \beta\right) - \sqrt{\ln c} + \beta + \frac{2}{n}\sqrt{\ln n}\right] \\
&\quad \geq \frac{1}{2\sqrt{\ln n}} \cdot \left(\sqrt{\ln c} - \frac{\ln c}{n} - \frac{1}{n} + \frac{1}{n^2} - \beta + \frac{\beta}{n} - \sqrt{\ln c} + \beta + \frac{2}{n}\sqrt{\ln n}\right) \\
&\quad \geq \frac{1}{2\sqrt{\ln n}}(-\sqrt{\ln c} - 1 + \beta + 2\sqrt{\ln n}).
\end{aligned}
$$

Since $\ln c < 2\ln n$, $n \geq 4$, and $\beta \geq 0.6$, we have

$$-\sqrt{\ln c} - 1 + \beta + 2\sqrt{\ln n} \geq (2 - \sqrt{2})\sqrt{\ln n} - 0.4 > 0.$$

PROOF OF LEMMA 3. Let us put $c = n^2/2m$, $c_1 = (n-1)^2/2(m - 2m\alpha/n)$. Then

$$
\begin{aligned}
&f(n-1; m - 2m\alpha/n; \beta) - f(n;m;\beta) \\
&\qquad \geq \frac{1}{2\sqrt{\ln n}}[(n-1)(\sqrt{\ln c_1} - \beta) - n(\sqrt{\ln c} - \beta)].
\end{aligned}
$$

Hence it is sufficient to show that $(n-1)\sqrt{\ln c_1} \geq n\sqrt{\ln c} - \beta$. From the inequality for α in the condition of the lemma it follows that

$$2\ln c - 2\beta\sqrt{\ln c} \leq 2(\alpha - 1) < 2(\alpha - 1)\cdot(n-1)^2/(n(n-2)).$$

Since $x < \ln(1-x)^{-1}$ for any $x \in (0,1)$, we have

$$
\begin{aligned}
2(\alpha-1)\cdot\frac{(n-1)^2}{n(n-2)} &\leq \frac{(n-1)^2}{n}\cdot\ln\left(1 - \frac{2(\alpha-1)}{n-2}\right)^{-1} = \frac{(n-1)^2}{n}\cdot\ln\frac{n-2}{n-2\alpha} \\
&\leq \frac{(n-1)^2}{n}\cdot\ln\frac{(1-1/n)^2}{1-2\alpha/n} = \frac{(n-1)^2}{n}\left(\ln\frac{n^2(1-1/n)^2}{2m(1-2\alpha/n)} - \ln\frac{n^2}{2m}\right) \\
&= \frac{(n-1)^2}{n}(\ln c_1 - \ln c),
\end{aligned}
$$

that is,

$$(n-1)^2(\ln c_1 - \ln c) \geq 2n\ln c - 2\beta n\sqrt{\ln c}.$$

Consequently,

$$
\begin{aligned}
(n-1)^2\ln c_1 &\geq n^2\ln c + \ln c - 2\beta n\sqrt{\ln c} \\
&= (n\sqrt{\ln c} - \beta)^2 + (\ln c - \beta^2) > (n\sqrt{\ln c} - \beta)^2,
\end{aligned}
$$

as required.

PROOF OF LEMMA 4. Putting $\alpha \doteq 1+\ln c-\beta\sqrt{\ln c}$ and $y \doteq (c-z)/(\alpha-z)$, we have

$$\varphi'(c;z;\beta) = -\sqrt{\ln\frac{c-z}{\alpha-z}}+\beta-(c-z)\frac{\dfrac{1}{c-z}-\dfrac{1}{\alpha-z}}{2\sqrt{\ln\dfrac{c-z}{\alpha-z}}}$$

$$= \frac{y-1-2\ln y+2\beta\sqrt{\ln y}}{2\sqrt{\ln y}}.$$

Since

$$y=\frac{c-z}{1+\ln c-\beta\sqrt{\ln c}-z}\geq\frac{c-1}{\ln c-\beta\sqrt{\ln c}},$$

by Lemma 1 for c, z and β that satisfy the condition of our lemma we have $y(c;z;\beta)\geq 2$. Hence, by Lemma 1 we have $\varphi'_z(c;z;\beta)>0$.

PROOF OF LEMMA 5. Putting

$$u\doteq\frac{c-z}{c(1+\ln c-\beta\sqrt{\ln c}-z)},$$

we have

$$\varphi'_c(c;z;\beta)=\sqrt{\ln\frac{c-z}{1+\ln c-\beta\sqrt{\ln c}-z}}$$

$$+\left(1-\frac{(c-z)(1-\beta/2\sqrt{\ln c})}{c(1+\ln c-\beta\sqrt{\ln c}-z)}\right)\Big/2\sqrt{\ln\frac{c-z}{1+\ln c-\beta\sqrt{\ln c}-z}}$$

$$-\sqrt{\ln c}-1/2\sqrt{\ln c}$$

$$=\sqrt{\ln cu}+\left(1-u\left(1-\frac{\beta}{2\sqrt{\ln c}}\right)\right)\Big/2\sqrt{\ln cu}-\sqrt{\ln c}-\frac{1}{2\sqrt{\ln c}}$$

$$=(\sqrt{\ln cu}-\sqrt{\ln c})\left(1-\frac{1}{2\sqrt{\ln c}\sqrt{\ln cu}}\right)-u\left(1-\frac{\beta}{2\sqrt{\ln c}}\right)\Big/2\sqrt{\ln cu}$$

$$=\frac{\ln u}{\sqrt{\ln cu}+\sqrt{\ln c}}\left(1-\frac{1}{2\sqrt{\ln c}\cdot\sqrt{\ln cu}}\right)-u\left(1-\frac{\beta}{2\sqrt{\ln c}}\right)\Big/2\sqrt{\ln cu}.$$

By Lemma 1, $cu>2$. Hence, $1-1/2\sqrt{\ln c}\sqrt{lncu}>0$. If $\ln u\leq 0$, then $\varphi'_c(c;z;\beta)<0$. Suppose that $u>1$. Then $\varphi'_c(c;z;\beta)<0$ is equivalent to

$$\frac{u}{\ln u}\left(1-\frac{\beta}{2\sqrt{\ln c}}\right)>\frac{2\sqrt{\ln cu}-1/\sqrt{\ln c}}{\sqrt{\ln cu}+\sqrt{\ln c}}. \qquad (**)$$

For any $u>1$ we have $u/\ln u\geq e$. By the condition of the lemma, $1-\beta/2\sqrt{\ln c}>1/2$. Hence, if the right side of (**) does not exceed $4/3<e/2$, then (**) is true. Suppose that

$$2\sqrt{\ln cu}-1/\sqrt{\ln c}>\frac{4}{3}(\sqrt{\ln c}+\sqrt{\ln cu}).$$

Then $\sqrt{\ln cu}>2\sqrt{\ln c}/3+3/2\sqrt{\ln c}$, $\ln cu>4\ln c/9+6+9/4\ln c$, and $\ln u>6$. But $u/\ln u>4$ for $u>e^6$, and since the right-hand side of (**) does not exceed 2, (**) is true.

PROOF OF LEMMA 6. For $x \geq 1.7$ we have that $g(x) \geq h(x) \doteq e^{1.1x}/4 + 1.1x - x^2 - 0.1$, and $h'(x) = 1.1e^{1.1x}/4 + 1.1 - 2x$ is a convex downwards function. Hence, h has no more than one minimum. We observe that

$$h'(2.3) = 1.1e^{2.53}/4 + 1.1 - 4.6 \leq 1.1 \cdot 12.56/4 - 3.5 < 3.46 - 3.5 < 0;$$

$$h'(2.34) = 1.1e^{2.574}/4 + 1.1 - 4.68 \geq 1.1 \cdot 13.118/4 - 3.58 > 3.6 - 3.58 > 0.$$

Consequently, it is sufficient to verify that $h(x) > 0$ for $x = 1.7$ and $x \in [2.3, 2.34]$. But $h(1.7) = e^{1.87}/4 + 1.87 - 2.89 - 0.1 \geq 6.488/4 - 1.12 > 0$. If $x \in [2.3, 2.34]$, then

$$\begin{aligned} h(x) &\geq e^{1.1 \cdot 2.3}/4 - (2.34^2 - 1.1 \cdot 2.34) - 0.1 \geq 12.552/4 - 2.34 \cdot 1.24 - 0.1 \\ &= 3.138 - 2.9016 - 0.1 > 0. \end{aligned}$$

Translated by E. J. F. PRIMROSE

Table

β	z	c	$\sqrt{\ln c} \le$	$1+\ln c-\beta\sqrt{\ln c}-z \le$	$\frac{c-z}{1+\ln c-\beta\sqrt{\ln c}-z} \ge$	$\sqrt{\ln \frac{c-z}{1+\ln c-\beta\sqrt{\ln c}-z}} \ge$	$\varphi(c;z;\beta) \ge$
2/3	1	3.15	1.072	0.4346	4.947	1.264	0.007
3/4	1	4.15	1.193	0.5285	5.96	1.336	0.0074
7/8	1	6.2	1.351	0.6431	8.085	1.445	0.012
1	1	9.4	1.497	0.7441	11.288	1.5567	0.004
1	1.1	10.1	1.521	0.69246	12.99	1.601	0.14
1	1.17	10.67	1.539	0.66	14.3939	1.633	0.26
1	1.2	11.2	1.555	0.6631	15.08	1.647	0.2
1	1.227	12	1.577	0.683	15.77	1.66	0.28
1	1.2545	13	1.602	0.71	16.54	1.674	0.09
1.1	1.45	17.99	1.7	0.57	29.017	1.835	1.3
0.6	1	2.55	0.968	0.3561	4.35	1.212	0.01