ELSEVIER

Discrete Mathematics 138 (1995) 57–62

DISCRETE MATHEMATICS

On set systems without weak 3-Δ-subsystems

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Received 20 July 1993; revised 14 January 1994

Abstract

A collection of sets is called a weak Δ-system if sizes of all pairwise intersections of these sets coincide. We prove a new upper bound on the function $f_w(n)$, the maximal size of a collection of n-element sets no three of which form a weak Δ-system. Namely, we prove that, for every $\delta > 0$, $f_w(n) = o(n!^{1/2+\delta})$.

We say that three sets A, B, C form a (3-)Δ-system if $A \cap B = B \cap C = C \cap A$. If a weaker condition $|A \cap B| = |B \cap C| = |C \cap A|$ is satisfied then these sets are said to form a weak (3-)Δ-system.

A long-standing question, widely advertised by Erdős, asks how large a collection of n-element sets can be if no three of its members form a Δ-system. Two problems are closely related with this one: its analogue for weak Δ-systems, and the problem of finding the maximal size $r(n)$ of a complete graph whose edges can be coloured by n colours without monochromatic triangles.

Denote by $f(n)$, resp. by $f_w(n)$, the maximal size of a collection of n-element sets no three of which form a Δ-system, resp. a weak Δ-system.

Note that the collection of N n-element sets without weak Δ-systems gives rise to a colouring of a complete graph on N vertices into n colours without monochromatic triangles; with elements of the system as vertices, and their intersection sizes as edge colours. Therefore $f_w(n) \leqslant r(n)$; and obviously $f_w(w) \leqslant f(n)$.

The problem of determining the exact growth rate of $f(n)$ was first raised in [4]. Since then it has become one of the most famous unsolved problems in combinatorics, and one of the favourite problems of Erdős. It attracted much attention, as well as

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[2] Supported by the grant 93-011-1486 of the Russian Fundamental Research Foundation.

SSDI 0012-365X(94)00185-5

similar problems on the growth rate of the functions $f_w(n)$ and $r(n)$, which are also unsolved. In all three cases, examples show that the functions grow at least exponentially. But, up to now, the best known upper bounds for them are hyperexponential.

Here are, to our knowledge, the best lower and upper bounds for these functions obtained so far, and the relevant references:

$c \cdot 10^{n/2} < f(n) < (1 + o(1))^n n!$ (the lower bound by Abbott and Hanson, the upper bound by Spencer; cf. [3]);

$$c \cdot 315^{n/5} < r(n) < (e - \tfrac{1}{24})n! \quad [7, \text{Theorem } 2.19];$$

$$c \cdot 5^{n/2} < f_w(n) < c'(n-d)! \quad \text{for any } d.$$

In the latter case, the lower bound is by Abbott [1]. We have been unable to find this upper bound in the literature; and Erdős [3] writes that he does not know if anybody proved that $f_w(n) < n!$; but it can easily be proved by a standard Ramsey-type argument. Indeed, in any collection of n-sets without weak 3-Δ-systems the sets intersecting a given one by at least $n - d$ elements mutually intersect by at least $n - 2d$ elements; hence their number does not exceed $r(2d)$. So we have inequality $f_w(n) \leqslant 1 + (n - d) f_w(n - 1) + r(2d)$, and the upper bound follows.

Erdős conjectures that all these functions are of exponential growth. But it is stated in [2, 6] that it is still not proved (and would be very desirable to prove) that $f(n) < Cn!$, and that $f_w(n) < Cn!^{1-\varepsilon}$ for some $\varepsilon > 0$.

Recently Kostochka [8] proved that, for any $C > 0$, $f(n) = o(n!/C^n)$.

In this paper we present a proof of the following theorem.

Theorem 1. *For any $\varepsilon < \frac{1}{2}$, there exists a constant C such that $f_w(n) < Cn!^{1-\varepsilon}$.*

Proof. Fix ε. The proof proceeds by induction on n. First we carry out the induction step, and make sure that the argument holds for n large enough, $n \geqslant n_0$, no matter what value of C is (in particular, the value n_0 does not depend on C). Then we choose

$$C > \max_{n \leqslant n_0} \frac{f_w(n)}{n!^{1-\varepsilon}},$$

thus providing the induction base.

So now we fix an arbitrary $C > 0$. Let $\mathscr{F}$ be a collection of n-sets without weak 3-Δ-subsystems such that $|\mathscr{F}| = f_w(n)$, and suppose that for $n' < n$ the inequality $f_w(n') < Cn'!^{1-\varepsilon}$ holds.

The following easy lemma will be used throughout the proof without further notice.

Lemma 1. *Let X be any subset of size $m < n$ of the ground set. Then the collection of $(n - m)$-sets*

$$\{F \setminus X \mid F \in \mathscr{F}, X \subset F\}$$

has no weak 3-Δ-subsystems.

Proof of Theorem 1 (*continued*). Choose real numbers α, L, and positive integers R, M so that the following inequalities be satisfied:

$$\alpha > \varepsilon + \frac{\varepsilon}{M-1}, \tag{1}$$

$$L > 2R, \tag{2}$$

$$\alpha < \varepsilon + \frac{1-\varepsilon}{R}, \tag{3}$$

$$\alpha > \varepsilon + \frac{M(1-\varepsilon)+2\varepsilon-1}{MR-1}. \tag{4}$$

For this to be possible, it is sufficient to choose M and R to satisfy the inequalities

$$\frac{\varepsilon}{M-1} < \frac{1-\varepsilon}{R}, \tag{5}$$

$$\frac{M(1-\varepsilon)+2\varepsilon-1}{MR-1} < \frac{1-\varepsilon}{R}, \tag{6}$$

Indeed, after this, the choice of α and L becomes easy.

Since $\varepsilon < \frac{1}{2}$, the inequality (6) is equivalent to

$$R > \frac{1-\varepsilon}{1-2\varepsilon};$$

and the inequality (5) can be rewritten as

$$M > 1 + \frac{R\varepsilon}{1-\varepsilon}.$$

Thus, we can satisfy the inequalities (1)–(4) by choosing first the value of R, then of M, and then of α and L.

Let $k = n^\alpha$, $l = Ln^\alpha$. We may suppose that k is integer.

The following lemma will be used in the proof to deal with intersections of comparatively large size.

Lemma 2. *For any* $A \in \mathscr{F}$,

$$\frac{k^k |\{X \in \mathscr{F} : |X \cap A| \geqslant Mk\}|}{Cn!^{1-\varepsilon}} \leqslant n^{o(n^\alpha)+zn^\alpha},$$

where $z = \varepsilon M - \alpha(M-1) < 0$.

Proof. Denote by P the quantity we want to estimate. By Lemma 1, we can apply the induction hypothesis to estimate the number of sets X with any given intersection

$A \cap X$ of size at least Mk:

$$|\{X \in \mathscr{F}: |X \cap A| \geqslant Mk\}| \leqslant \sum_{i \geqslant Mk} \binom{n}{i} C(n-i)!^{1-\varepsilon}.$$

Thus,

$$P \leqslant n^{\alpha n^{\alpha}} \cdot n!^{-1+\varepsilon} \cdot n \cdot \max_{i \geqslant Mk} \binom{n}{i}(n-i)!^{1-\varepsilon}.$$

To determine the value of i at which the maximum is attained, consider the ratio

$$\frac{\binom{n}{i+1}(n-i-1)!^{1-\varepsilon}}{\binom{n}{i}(n-i)!^{1-\varepsilon}} = \frac{n-i}{(i+1)(n-i)^{1-\varepsilon}} < \frac{n^{\varepsilon}}{Mn^{\alpha}} < 1,$$

since $\varepsilon < \alpha$. So, the maximum is attained at $i = Mn^{\alpha}$, and we have

$$\begin{aligned} P &\leqslant n^{o(n^{\alpha})} n^{\alpha n^{\alpha}} n^{(1-\alpha)Mn^{\alpha}} n^{(-1+\varepsilon)Mn^{\alpha}} \\ &= n^{o(n^{\alpha})} n^{n^{\alpha}(\alpha + M(1-\alpha) + M(-1+\varepsilon))} \\ &= n^{o(n^{\alpha})} n^{zn^{\alpha}}. \end{aligned}$$

The inequality (1) asserts that $z < 0$, so the lemma is proved. □

Proof of Theorem 1 (*continued*). Now we proceed to the proof of the induction step.

Contruct inductively a sequence $\mathscr{F}_0, \dots, \mathscr{F}_m$ of subcollections of $\mathscr{F}$ and a sequence $I_0, \dots, I_m$ of subsets of $I = \{0, \dots, k-1\}$ by the following rules:

$$\mathscr{F}_0 = \mathscr{F}, \quad I_0 = I.$$

For $i = 0, 1, \dots,$ if one can find a set $F_i \in \mathscr{F}_i$ and a number $x_i \in I_i$ such that

$$|\{X \in \mathscr{F}_i: |X \cap F_i| = x_i\}| \geqslant \frac{|\mathscr{F}_i|}{l}$$

then let

$$I_{i+1} = I_i \setminus \{x_i\}; \quad \mathscr{F}_{i+1} = \{X \in \mathscr{F}_i: |X \cap F_i| = x_i\}.$$

Otherwise stop; let $\chi = \mathscr{F}_i$.

Note that intersection sizes excluded from I during this process cannot appear in the collection χ. Indeed, if we had $|X \cap Y| = x_j$ for some $X, Y \in \chi$ then the sets F_j, X, Y would form a weak 3-Δ-system. This is the only place in the proof in which absence of weak 3-Δ-systems is used at its full strength.

The process stops after at most k steps; we have

$$|\chi| \geqslant \frac{|\mathscr{F}|}{l^k} = \frac{|\mathscr{F}|}{L^k k^k}. \tag{7}$$

Also we have that at most $|\chi| k/l = |\chi|/L$ sets from χ intersect any given $A \in \chi$ by less than k elements.

Now choose R sets $A_1, \ldots, A_R \in \chi$ such that $|A_i \cap A_j| < Mk$ for all $i \neq j$. We can choose them one by one, starting with an arbitrary A_1. If, at a certain moment, there is no appropriate set A_j then it means that

$$\chi \subseteq \bigcup_{i=1}^{j-1} \{X: |X \cap A_i| \geqslant Mk\}.$$

But then Lemma 2 together with the inequality (7) imply

$$|\mathscr{F}| \leqslant L^k R n^{o(n^z)+zn^z} Cn!^{1-\varepsilon} < Cn!^{1-\varepsilon}$$

and the induction step is proved.

So we suppose that the sets A_i are chosen. Now we will use them to estimate $|\chi|$. To this end, we partition χ into three collections χ_0, χ_1, χ_2 and deal with them separately.

Let

$$\chi_0 = \{X \in \chi: |X \cap A_i| < k \text{ for some } i\};$$

$$\chi_2 = \{X \in \chi: |X \cap A_i| \geqslant Mk \text{ for some } i\};$$

$$\chi_1 = \{X \in \chi: k \leqslant |X \cap A_i| < Mk \text{ for all } i\}.$$

We have $|\mathscr{F}| \leqslant L^k \cdot k^k(|\chi_0| + |\chi_1| + |\chi_2|)$.

It is easy to deal with χ_0 and χ_2. Indeed,

$$|\chi_0| \leqslant R|\chi|/L < |\chi|/2 \quad \text{(from the inequality (2));}$$

$$L^k k^k |\chi_2| \leqslant L^{n^z} n^{o(n^z)+zn^z} Cn!^{1-\varepsilon} = o(Cn!^{1-\varepsilon}) \quad \text{(by Lemma 2).}$$

We shall estimate $|\chi_1|$ by considering all possible intersections of $A_1, \ldots, A_R$ with sets from this collection.

Let $B = \bigcup_{i \neq j}(A_i \cap A_j)$; $A_i' = A_i \backslash B$. We have $|B| \leqslant R^2 Mk$, and $|A_i'| \leqslant n$.

For each $X \in \chi_1$, define $b = b(X) = |X \cap B|$, $a_i = a_i(X) = |X \cap A_i'|$. These numbers obviously satisfy the inequalities $0 \leqslant a_i < Mk$, and $b + a_i \geqslant k$.

Given $b, a_1, \ldots, a_R$, let us estimate the number of sets X with such parameters. This number does not exceed

$$N = \binom{|B|}{b}\binom{n}{a_1}\cdots\binom{n}{a_R} C(n - a_1 - \cdots - a_R - b)!^{1-\varepsilon}.$$

Let $s = \lceil \sum a_i/R \rceil$; $\min a_i \leqslant s \leqslant Mk$.

From the inequalities $a_1 + \cdots + a_R \geqslant Rs - R$, $b + \min a_i \geqslant k$, and $\min a_i \leqslant s$ we find that

$$n - a_1 - \cdots - a_R - b \leqslant n - k - Rs + s + R.$$

The factor $\binom{n}{a_1}\cdots\binom{n}{a_R}$ is bounded from above by $\binom{n}{s}^R$. So, we have

$$N \leqslant 2^{R^2 M n^z} \binom{n}{s}^R C(n - k - Rs + s + R)!^{1-\varepsilon}$$

$$\leqslant n^{o(n^z)} \phi(s),$$

where $\phi(s) = C\binom{n}{s}^R (n - (R-1)s - k)!^{1-\varepsilon}$.

As in the proof of Lemma 2, we shall now determine the maximum value of $\phi(s)$ when $0 \leqslant s \leqslant Mk$.

$$\frac{\phi(s+1)}{\phi(s)} = \left(\frac{n-s}{s+1}\right)^R \left(\frac{(n-k-(R-1)s-(R-1))!}{(n-k-(R-1)s)!}\right)^{1-\varepsilon}$$

$$= (1+\mathrm{o}(1))\left(\frac{n}{s+1}\right)^R n^{(1-R)(1-\varepsilon)}$$

$$\geqslant (1+\mathrm{o}(1))n^{(1-\alpha)R} n^{(1-R)(1-\varepsilon)} > 1,$$

since $(1-\alpha)R + (1-R)(1-\varepsilon) > 0$ (inequality (3)).

Thus the maximum is attained at $s = Mk$, and we have

$$|\chi_1| \leqslant (|B|+1)(|A'_1|+1)\cdots(|A'_R|+1)n^{\mathrm{o}(n^\alpha)}\phi(Mk)$$

$$= n^{\mathrm{o}(n^\alpha)} C \left(\frac{n}{Mn^\alpha}\right)^R (n - n^\alpha(MR - M + 1))!^{1-\varepsilon};$$

$$\frac{L^k k^k |\chi_1|}{Cn!^{1-\varepsilon}} = n^{\mathrm{o}(n^\alpha)} \left(\frac{n}{Mn^\alpha}\right)^{MRn^\alpha} n^{-n^\alpha(MR-M+1)(1-\varepsilon)} n^{\alpha n^\alpha}$$

$$= n^{\mathrm{o}(n^\alpha)} n^{n^\alpha(MR(1-\alpha)+\alpha-(MR-M+1)(1-\varepsilon))} = \mathrm{o}(1),$$

by the inequality (4).

Combining these estimates for $|\chi_0|$, $|\chi_1|$, and $|\chi_2|$, we get that $|\mathscr{F}| = \mathrm{o}(Cn!^{1-\varepsilon})$ and thus prove the induction step. The theorem is proved. $\square$

Acknowledgement

The authors are very grateful to anonymous referees for their useful remarks and suggestions.

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