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A BROOKS-TYPE RESULT FOR SPARSE CRITICAL GRAPHS

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A graph G is *k-critical* if it has chromatic number k, but every proper subgraph of G is $(k-1)$-colorable. Let $f_k(n)$ denote the minimum number of edges in an n-vertex k-critical graph. Recently the authors gave a lower bound, $f_k(n) \geq \left\lceil \frac{(k+1)(k-2)|V(G)|-k(k-3)}{2(k-1)} \right\rceil$, that solves a conjecture by Gallai from 1963 and is sharp for every $n \equiv 1 \pmod{k-1}$. It is also sharp for $k=4$ and every $n \geq 6$. In this paper we refine the result by describing all n-vertex k-critical graphs G with $|E(G)| = \frac{(k+1)(k-2)|V(G)|-k(k-3)}{2(k-1)}$. In particular, this result implies exact values of $f_5(n)$ for $n \geq 7$.

1. Introduction

A *proper k-coloring*, or simply *k-coloring*, of a graph $G=(V,E)$ is a function $f\colon V \to \{1,2,\dots,k\}$ such that for each $uv \in E$, $f(u) \neq f(v)$. A graph G is *k-colorable* if there exists a k-coloring of G. The *chromatic number*, $\chi(G)$, of a graph G is the smallest k such that G is k-colorable. A graph G is *k-chromatic* if $\chi(G)=k$.

A graph G is *k-critical* if G is k-chromatic, but every proper subgraph of G is $(k-1)$-colorable. Critical graphs were first defined and used by Dirac [7,8,9] in 1951-52. A reason to study k-critical graphs is that every

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k-chromatic graph contains a k-critical subgraph and k-critical graphs have more restricted structure. For example, k-critical graphs are 2-connected and $(k-1)$-edge-connected.

One of the basic questions on k-critical graphs is: *What is the minimum number $f_k(n)$ of edges in a k-critical graph with n vertices?* This question was first asked by Dirac [12] in 1957 and then was reiterated by Gallai [17] in 1963, Ore [29] in 1967 and others [21,22,34]. Gallai [17] has found the values of $f_k(n)$ for $n \le 2k-1$.

Theorem 1 (Gallai [17]). *If $k \ge 4$ and $k+2 \le n \le 2k-1$, then*

$$f_k(n) = \frac{1}{2}\left((k-1)n + (n-k)(2k-n)\right) - 1.$$

Kostochka and Stiebitz [24] found the value $f_k(2k) = k^2 - 3$. Gallai [16] also conjectured the exact value for $f_k(n)$ for $n \equiv 1 \pmod{k-1}$.

Conjecture 2 (Gallai [16]). *If $k \ge 4$ and $n \equiv 1 \pmod{k-1}$, then*

$$f_k(n) = \frac{(k+1)(k-2)n - k(k-3)}{2(k-1)}.$$

The upper bound on $f_k(n)$ follows from Gallai's construction of k-critical graphs with only one vertex of degree at least k. So the main difficulty of the conjecture is in proving the lower bound on f_k.

For a graph G and a vertex $u \in V(G)$, a *split* of u is a construction of a new graph G' such that $V(G') = V(G) - u + \{u', u''\}$, where $G - u \cong G' - \{u', u''\}$, $N(u') \cup N(u'') = N(u)$, and $N(u') \cap N(u'') = \emptyset$. A *DHGO-composition* $O(G_1, G_2)$ of graphs G_1 and G_2 is a graph obtained as follows: delete some edge xy from G_1, split some vertex z of G_2 into two vertices z_1 and z_2 of positive degree, and identify x with z_1 and y with z_2. Note that DHGO-composition could be found in Dirac's paper [13] and has roots in [10]. It was also used by Gallai [16] and Hajós [19]. Ore [29] used it for a composition of complete graphs.

The mentioned authors observed that if G_1 and G_2 are k-critical and G_2 is not k-critical after z has been split, then $O(G_1, G_2)$ also is k-critical. This observation implies

$$(1)\quad f_k(n+k-1) \le f_k(n) + \frac{(k+1)(k-2)}{2} = f_k(n) + (k-1)\frac{(k+1)(k-2)}{2(k-1)}.$$

Ore believed that this construction starting from an extremal graph on at most $2k$ vertices with $G_2 = K_k$ at each iteration yields sparsest k-critical n-vertex graphs for all $n \ge 2k$.

Conjecture 3 (Ore [29]). *If $k \geq 4$, $n \geq k$ and $n \neq k+1$, then $f_k(n+k-1) = f_k(n) + (k-2)(k+1)/2$.*

Note that Conjecture 2 is equivalent to the case $n \equiv 1 \pmod{k-1}$ of Conjecture 3.

Some lower bounds on $f_k(n)$ were obtained in [12,28,16,24,25,15]. Recently, the authors [26] proved Conjecture 2 valid.

Theorem 4 ([26]). *If $k \geq 4$ and G is k-critical, then $|E(G)| \geq \left\lceil \frac{(k+1)(k-2)|V(G)|-k(k-3)}{2(k-1)} \right\rceil$. In other words, if $k \geq 4$ and $n \geq k$, $n \neq k+1$, then*

$$f_k(n) \geq F(k,n) := \left\lceil \frac{(k+1)(k-2)n - k(k-3)}{2(k-1)} \right\rceil .$$

The result also confirms Conjecture 3 in several cases.

Corollary 5 ([26]). *Conjecture 3 is true if*

(i) $k=4$,
(ii) $k=5$ *and* $n \equiv 2 \pmod 4$, *or*
(iii) $n \equiv 1 \pmod{k-1}$.

Some applications of Theorem 4 are given in [26] and [5]. In [27], the authors derive from a partial case of Theorem 4 a half-page proof of the well-known Grötzsch Theorem [18] that every planar triangle-free graph is 3-colorable. Conjecture 3 is still open in general. By examining known values of $f_k(n)$ when $n \leq 2k$, it follows that $f_k(n) - F(k,n) \leq k^2/8$.

The goal of this paper is to describe the k-*extremal* graphs, i.e., the k-critical graphs G such that $|E(G)| = \frac{(k+1)(k-2)|V(G)|-k(k-3)}{2(k-1)}$. This is a refinement of Conjecture 2: For $n \equiv 1 \pmod{k-1}$, we describe all n-vertex k-critical graphs G with $|E(G)| = f_k(n)$. This is also the next step towards the full solution of Conjecture 3.

By definition, if G is k-extremal, then $\frac{(k+1)(k-2)|V(G)|-k(k-3)}{2(k-1)}$ is an integer, and so $|V(G)| \equiv 1 \pmod{k-1}$. For example, K_k is k-extremal.

Suppose that G_1 and G_2 are k-extremal and $G = O(G_1, G_2)$. Then

$$\begin{aligned} |E(G)| &= |E(G_1)| + |E(G_2)| - 1 \\ &= \frac{(k+1)(k-2)(|V(G_1)| + |V(G_2)|) - 2k(k-3)}{2(k-1)} - 1 \\ &= \frac{(k+1)(k-2)|V(G)| - k(k-3)}{2(k-1)}. \end{aligned}$$

After z is split, G_2 will still have $F(k,|V(G_2)|) < F(k,|V(G_2)|+1)$ edges, and therefore will not be k-critical. Thus the DHGO-composition of any two k-extremal graphs is again k-extremal.

A graph is a *k-Ore graph* if it is obtained from a set of copies of K_k by a sequence of DHGO-compositions. By the above, every k-Ore graph is k-extremal. This yields an explicit construction of infinitely many k-extremal graphs.

The main result of the present paper is the following.

Theorem 6. *Let $k \geq 4$ and G be a k-critical graph. Then G is k-extremal if and only if it is a k-Ore graph. Moreover, if G is not a k-Ore graph, then $|E(G)| \geq \frac{(k+1)(k-2)|V(G)|-y_k}{2(k-1)}$, where $y_k = \max\{2k-6, k^2-5k+2\}$. Thus $y_4 = 2$, $y_5 = 4$, and $y_k = k^2-5k+2$ for $k \geq 6$.*

The message of Theorem 6 is that although for every $k \geq 4$ there are infinitely many k-extremal graphs, they all have a simple structure. In particular, every k-extremal graph distinct from K_k has a separating set of size 2. The theorem gives a slightly better approximation for $f_k(n)$ and adds new cases for which we now know the exact values of $f_k(n)$:

Corollary 7. *Conjecture 3 holds and the value of $f_k(n)$ is known if*

(i) $k \in \{4,5\}$,
(ii) $k = 6$ *and* $n \equiv 0 \pmod 5$,
(iii) $k = 6$ *and* $n \equiv 2 \pmod 5$,
(iv) $k = 7$ *and* $n \equiv 2 \pmod 6$, *or*
(v) $k \geq 4$ *and* $n \equiv 1 \pmod{k-1}$.

This value of y_k in Theorem 6 is best possible in the sense that for every $k \geq 4$, there exist infinitely many 3-connected graphs G with $|E(G)| = \frac{(k+1)(k-2)|V(G)|-y_k}{2(k-1)}$. The idea of this construction (Construction 58) and the examples for $k = 4, 5$ are due to Toft ([33], based on [32]). Construction 60 produces the examples for $k \geq 6$. Theorem 6 has already found interesting applications. In [3], it was used to describe the 4-critical planar graphs with exactly 4 triangles. This problem was studied by Axenov [1] in the seventies, and then mentioned by Steinberg [31] (quoting Erdős from 1990), and Borodin [2]. It was proved in [3] that the 4-critical planar graphs with exactly 4 triangles and no 4-faces are exactly the 4-Ore graphs with exactly 4 triangles. Also, Kierstead and Rabern [23] and independently Postle [30] have used Theorem 6 to describe the infinite family of 4-critical graphs G with the property that for each edge $xy \in E(G)$, $d(x)+d(y) \leq 7$. It turned out that such graphs form a subfamily of the family of 4-Ore graphs.

Our proofs will use the language of *potentials*.

Definition 8. Let G be a graph. For $R \subseteq V(G)$, define *the k-potential of R* to be

$$\rho_{k,G}(R) = (k+1)(k-2)|R| - 2(k-1)|E(G[R])|. \tag{2}$$

When there is no chance for confusion, we will use $\rho_k(R)$. Let $P_k(G) = \min_{\emptyset \neq R \subseteq V(G)} \rho_k(R)$.

Informally, $\rho_{k,G}(R)$ measures how many edges are needed to be added to $G[R]$ (or removed, if the potential is negative) in order to obtain a graph with average degree $\frac{(k+1)(k-2)}{k-1}$. Our proofs below will involve adding and deleting edges and vertices, so using the language of potentials helps keep track of whether or not the resulting graph maintains the assumptions of the theorem.

Translated into the language of potentials, Theorem 4 sounds as follows.

Corollary 9 ([26]). *If G is k-critical, then $\rho_k(V(G)) \leq k(k-3)$. In particular, if a graph H satisfies $\rho_{k,H}(S) > k(k-3)$ for all nonempty $S \subseteq V(H)$, then H is $(k-1)$-colorable.*

Similarly, our main result, Theorem 6, is:

Theorem 10. *If G is k-critical and not a k-Ore graph, then*

$$\rho_k(V(G)) \leq y_k,$$

where $y_k = \max\{2k-6, k^2-5k+2\}$. In particular, if a graph H does not contain a k-Ore graph as a subgraph and $\rho_{k,H}(S) > y_k$ for all nonempty $S \subseteq V(H)$, then H is $(k-1)$-colorable.

Our strategy of the proof (similar to those in [4,6,26,27]) is to consider a minimum counter-example J to Theorem 10 and derive a set of its properties leading to a contradiction. Quite useful claims will be that all nontrivial proper subsets of $V(J)$ have "high" potentials. Important examples of such claims are Claim 27 and Lemma 38 below. This will help us to provide $(k-1)$-colorings of subgraphs of J with additional properties. For example, Claim 27 will imply Claim 28 that adding any edge to a subgraph H of J with $1 < |V(H)| < |V(J)|$ leaves the subgraph $(k-1)$-colorable. Important new ingredient of the proof is the study in the next section of the properties of k-Ore graphs and their colorings. In Section 3 we prove basic properties of our minimum counter-example J, including Claim 27 mentioned above. Then in Section 4 we introduce and study properties of *clusters* – sets of vertices of degree $k-1$ in J with the same closed neighborhood. This will allow us to prove Lemma 38. Based on this lemma and its corollaries, we prove Theorem 10 in Section 5 using some variations of discharging; the cases of small k will need separate considerations. In Section 6 we discuss the sharpness of our result and in Section 7 – some algorithmic aspects of it.

2. Potentials and Ore graphs

The fact below summarizes useful properties of ρ_k and y_k following directly from the definitions or Corollary 9.

Fact 11. *For the k-potential defined by (2), we have*

1. *Potential is submodular:*

(3) $\rho_k(X\cap Y)+\rho_k(X\cup Y) = \rho_k(X)+\rho_k(Y)-2(k-1)|E_G[X-Y,Y-X]|.$

2. $\rho_k(V(K_1))=(k+1)(k-2)$.
3. $\rho_k(V(K_2))=2(k^2-2k-1)$.
4. $\rho_k(V(K_{k-1}))=2(k-2)(k-1)$.
5. $\rho_k(V(K_k))=k(k-3)$.
6. *If $k\geq 4$, then $\rho_k(V(K_k)) \leq \rho_k(V(K_1)) \leq \rho_k(V(K_{k-1})) \leq \rho_k(V(K_2)) \leq \rho_k(V(K_i))$ for all $3\leq i\leq k-2$.*
7. *For any vertex set S, $\rho_k(S)\geq \rho_k(K_{|S|})$. In particular, if $1\leq |S|\leq k-1$, then $\rho_k(S)\geq (k+1)(k-2)$. If $2\leq |S|\leq k-1$, then $\rho_{k,G}(S)\geq 2(k-2)(k-1)$ with equality only if $|S|=k-1$ and $G[S]=K_{k-1}$.*
8. $k(k-3)\leq y_k+2k-2<(k+1)(k-2)$.
9. *$\rho_k(A)$ is even for each k and A.*
10. *If G is a graph with a spanning subgraph H that is a k-Ore graph, then $\rho_{k,G}(V(G))\leq k(k-3)$. If $H=G$, then we have equality. If H is a proper subgraph of G, then $\rho_{k,G}(V(G))\leq y_k$.*

A common technique in constructing critical graphs (see [21,31]) is to use *quasi-edges* and *quasi-vertices*. For $k\geq 3$, a graph G, and $x,y\in V(G)$, a *k-quasi-xy-edge* $Q_k(x,y)$ is a subset Q of $V(G)$ such that $x,y\in Q$ and

(Q1) $G[Q]$ has a $(k-1)$-coloring,
(Q2) $\phi(x)\neq\phi(y)$ for every proper $(k-1)$-coloring of $G[Q]$, and
(Q3) for any edge $e\in G[Q]$, $G[Q]-e$ has a $(k-1)$-coloring ϕ such that $\phi(x)=\phi(y)$.

Symmetrically, a *k-quasi-xy-vertex* $Q'_k(x,y)$ is a subset Q' of $V(G)$ such that $x,y\in Q'$ and

(Q′1) $G[Q']$ has a $(k-1)$-coloring,
(Q′2) $\phi(x)=\phi(y)$ for every proper $(k-1)$-coloring of $G[Q']$, and

(Q′3) for any edge $e \in G[Q']$, $G[Q'] - e$ has a $(k-1)$-coloring ϕ such that $\phi(x) \neq \phi(y)$.

If G is a k-critical graph, then for each $e = xy \in E(G)$, graph $G - e$ is a k-quasi-xy-vertex. On the other hand, given some k-quasi-vertices and k-quasi-edges, one can construct from copies of them infinitely many k-critical graphs. In particular, the DHGO-composition can be viewed in this way. The next observation is well known and almost trivial, but we state it, because we use it often.

Fact 12. *Let $k \geq 4$. If a k-critical graph G has a separating set $\{x,y\}$, then*

(1) $G - \{x,y\}$ has exactly two components, say with vertex sets A' and B';
(2) $xy \notin E(G)$;
(3) one of $A' \cup \{x,y\}$ and $B' \cup \{x,y\}$ is a k-quasi-xy-edge and the other is a k-quasi-xy-vertex.

A quasi-edge and a quasi-vertex are very related structures, as seen by the following construction.

Fact 13. *If $Q_k(x,z)$ is a k-quasi-xz-vertex and $Q'(x,y)$ is obtained from $Q_k(x,z)$ by appending a leaf y that is adjacent only to z, then $Q'(x,y)$ is a k-quasi-xy-edge. If $Q'_k(x,y)$ is a k-quasi-xy-edge and $N(y) \cap Q'_k(x,y) = \{z\}$, then the vertex set $Q_k(x,z) = Q'_k(x,y) - y$ is a k-quasi-xz-vertex.*

Fact 12 together with the definition of k-Ore graphs, implies the following.

Fact 14. *Every k-Ore graph $G \neq K_k$ has a separating set $\{x,y\}$ and two vertex subsets $A = A(G,x,y)$ and $B = B(G,x,y)$ such that*

(i) $A \cap B = \{x,y\}$, $A \cup B = V(G)$ and no edge of G connects $A - x - y$ with $B - x - y$,
(ii) the graph $\widetilde{G}(x,y)$ obtained from $G[A]$ by adding edge xy is a k-Ore graph,
(iii) the graph $\check{G}(x,y)$ obtained from $G[B]$ by gluing x with y into a new vertex $x * y$ is a k-Ore graph, and
(iv) $xy \notin E(G)$.

In terms of Fact 14, G is the DHGO-composition of $\widetilde{G}(x,y)$ and $\check{G}(x,y)$. We will repeatedly use the notation in this fact. The next fact directly follows from the definitions.

Fact 15. *Using the notation in Fact 14, we have*

1. A is a k-quasi-xy-vertex;

2. *B is a k-quasi-xy-edge;*
3. $\rho_{k,G}(A) = \rho_{k,K_1}(V(K_1)) = (k+1)(k-2)$;
4. $\rho_{k,G}(B) = \rho_{k,K_2}(V(K_2)) = 2(k^2-2k-1)$;
5. $N(x) \cap B \cap N(y) = \emptyset$;
6. $N_{\widetilde{G}}(v) = N_G(v)$ *for each* $v \in A - x - y$;
7. $d_{\check{G}}(v) = d_G(v)$ *for each* $v \in B - x - y$;
8. • *If* $R \subseteq A - x$ *or* $R \subseteq A - y$, *then* $\rho_{k,G}(R) = \rho_{k,\widetilde{G}}(R)$.
   • *If* $R \subseteq B - \{x,y\}$, *then* $\rho_{k,G}(R) = \rho_{k,\check{G}}(R)$.
   • *If* $R \subseteq B - x$ *or* $R \subseteq B - y$ *and* $\check{R} = R - \{x,y\} + x*y$, *then* $\rho_{k,G}(R) \geq \rho_{k,\check{G}}(\check{R})$.
   • *If* $\{x,y\} \subseteq R \subseteq B$ *and* $\check{R} = R - \{x,y\} + x*y$, *then* $\rho_{k,G}(R) = \rho_{k,\check{G}}(\check{R}) + (k+1)(k-2)$.
9. $\rho_{k,G}(V(G)) = \rho_{k,V(\widetilde{G})}(V(\widetilde{G})) = \rho_{k,V(\check{G})}(V(\check{G})) = k(k-3)$.

Claim 16. *For every k-Ore graph G and each $\emptyset \neq R \subsetneq V(G)$, we have $\rho_{k,G}(R) \geq (k+1)(k-2)$.*

Proof. Let G be a smallest counter-example to the claim and let

$$R \subsetneq V(G) \text{ be a smallest nonempty proper subset of } V(G) \text{ with } \rho_{k,G}(R) < (k+1)(k-2). \tag{4}$$

By Fact 11.7, $G \neq K_k$. So, by Fact 14 there is a separating set $\{x,y\}$ and two vertex subsets $A = A(G,x,y)$ and $B = B(G,x,y)$ as in Fact 14. By the minimality of G, every proper subset of $V(\widetilde{G}(x,y))$ and of $V(\check{G}(x,y))$ has potential at least $(k+1)(k-2)$. If $G[R]$ were disconnected, then the vertex set of some component of $G[R]$ would also have potential less than $(k+1)(k-2)$, contradicting the minimality of R. So, $G[R]$ is connected. Since $\rho_{k,G}(R) < (k+1)(k-2)$ by (4) and R is non-empty, by Fact 11.7, $|R| \geq k$.

Case 1: $\{x,y\} \cap R = \emptyset$. Since $G[R]$ is connected, R is a non-empty proper subset either of A or B. This contradicts Fact 15 and the minimality of G.

Case 2: $\{x,y\} \cap R = \{x\}$. The set $R \cap A$ induces a non-empty connected subgraph of G, and so by the minimality of $|R|$, $\rho_{k,G}(R \cap A) \geq (k+1)(k-2)$. Similarly, $\rho_{k,G}(R \cap B) \geq (k+1)(k-2)$. By Fact 11.1,

$$\rho_{k,G}(R) = \rho_{k,G}(R \cap A) + \rho_{k,G}(R \cap B) - \rho_{k,G}(\{x\}) \geq (k+1)(k-2),$$

a contradiction to (4).

Case 3: $\{x,y\}\subseteq R$. If $A\subseteq R$, then by Facts 11.1 and 15.3,

$$\rho_{k,\check{G}(x,y)}((R-A)+x*y) = \rho_{k,G}(R)-\rho_{k,G}(A)+\rho_{k,\check{G}(x,y)}(\{x*y\}) = \rho_{k,G}(R).$$

But by the minimality of G, this is at least $(k+1)(k-2)$, a contradiction to (4). Similarly, if $B\subseteq R$, then

$$\rho_{k,\widetilde{G}(x,y)}(R\cap A) = \rho_{k,G}(R) - \rho_{k,G}(B) + \rho_{k,\widetilde{G}(x,y)}(\{x,y\}) = \rho_{k,G}(R),$$

a contradiction to (4) again. So, suppose $A-R\neq\emptyset$ and $B-R\neq\emptyset$. By the minimality of G, we have $\rho_{k,\widetilde{G}(x,y)}(R\cap A)\geq(k+1)(k-2)$. Since xy is an edge in $\widetilde{G}(x,y)$ but not in G, this yields $\rho_{k,G}(R\cap A)\geq(k+1)(k-2)+2(k-1)$. Similarly, $\rho_{k,\check{G}(x,y)}((R-A)+x*y)\geq(k+1)(k-2)$ and thus $\rho_{k,G}(R\cap B)\geq 2(k+1)(k-2)$. Then

$$\rho_{k,G}(R) = \rho_{k,G}(R\cap A)+\rho_{k,G}(R\cap B)-2\rho_{k,G}(K_1) \geq (k+1)(k-2)+2(k-1),$$

a contradiction to (4). ∎

Definition 17. For a set U of vertices in a graph G, the *border* of U is $U_* = \{w\in U:\ N(w)\not\subseteq U\}$, i.e., the set of vertices in U that have neighbors outside of U.

If U_* is the border of $U\subset V(G)$ and $U_*\neq U$, then U_* is a separating set in G.

A set S of vertices in a graph G is *standard*, if

(a) $\rho_{k,G}(S)=(k+1)(k-2)$ and
(b) the border of S is a 2-element set $\{x,y\}$ such that $G[S-\{x,y\}]$ is connected, and
(c) S is a k-quasi-$\{x,y\}$-vertex.

Note that a standard set is a k-quasi-vertex whose k-potential is the same as that of a vertex.

Lemma 18. *Let G be a k-Ore graph. Let $W\subset V(G)$ with $|W|\geq 2$ and $\rho_k(W)\leq(k+1)(k-2)$. Then $G[W]$ is connected and contains a standard set.*

Proof. Suppose $\rho_k(W)\leq(k+1)(k-2)$. Then by Claim 16, $\rho_k(W)=(k+1)(k-2)$. If $G[W]$ is disconnected, say $W=W_1\cup W_2$ with $W_1\cap W_2=\emptyset$ and no edges between W_1 and W_2, then $\rho_{k,G}(W_1)+\rho_{k,G}(W_2)=\rho_{k,G}(W)$. This implies $\min\{\rho_{k,G}(W_1),\rho_{k,G}(W_2)\}\leq(k+1)(k-2)/2$, which contradicts Claim 16. So $G[W]$ is connected, i.e., the first part of the lemma holds.

To prove the second part, choose a counter-example G with the fewest vertices and let $W \subseteq V(G)$ be a smallest subset of $V(G)$ such that

$$|W| \geq 2, \rho_{k,G}(W) = (k+1)(k-2), \quad \text{and } W \text{ does not contain a standard set.} \tag{5}$$

By Fact 11.7, the graph K_k simply does not have sets W satisfying (5). So $G \neq K_k$ and thus by Fact 14 has a separating set $\{x,y\}$. Let A, B, $\widetilde{G}(x,y)$, and $\breve{G}(x,y)$ be defined as in Fact 14. First we show that

$$G[W] \text{ is 2-connected.} \tag{6}$$

Indeed, suppose not. Then by the first part of the lemma, $G[W]$ has a cut vertex, say z. Let W_1 and W_2 be two subsets of W such that $W_1 \cap W_2 = \{z\}$, $W_1 \cup W_2 = W$ and there are no edges between $W_1 - z$ and $W_2 - z$. Then by Fact 11.1 and Fact 11.2,

$$\rho_{k,G}(W_1) + \rho_{k,G}(W_2) = \rho_{k,G}(W) + \rho_{k,G}(\{z\}) = 2(k+1)(k-2).$$

So by Claim 16, $\rho_{k,G}(W_1) = \rho_{k,G}(W_2) = (k+1)(k-2)$. Thus, by the minimality of W, each of W_1 and W_2 contains a standard subset, a contradiction to (5). This proves (6).

Let $W_A = A \cap W$ and $W_B = B \cap W$. Suppose $S \subseteq W_A$, $\rho_{k,G}(S) = (k+1)(k-2)$ (which implies $S \neq \emptyset$), and $y \notin S$. Because (a) by Fact 15.8, S has the same potential in $\widetilde{G}$ as in G, (b) by Fact 14.ii, $\widetilde{G}$ is also k-Ore, and (c) G is a minimal counterexample,

$$S \text{ contains a standard set } W' \text{ in } \widetilde{G}(x,y). \tag{7}$$

We will use (7) in Cases 1 and 3 below.

Case 1: $W \subseteq A$. If $\{x,y\} \subseteq W$, then $\rho_{k,\widetilde{G}(x,y)}(W) = \rho_{k,G}(W) - 2(k-1) = k(k-3)$, which by Claim 16 means that $W = A$. But A is a standard set, a contradiction to the choice of W. So by symmetry, we may assume that $y \notin W$. By (7) with $S = W$, the set W contains a standard set W' in $\widetilde{G}(x,y)$. Since $y \notin W'$, by Fact 15.8, W' has the same potential in G as in $\widetilde{G}$. So by the minimality of W, $W' = W$. Furthermore, if $x \in W'$, then it is in the border of W' in $\widetilde{G}$ because $y \notin W = W'$ and $xy \in E(\widetilde{G})$. By this and Fact 15.6, we conclude that the border of W' in $\widetilde{G}$ coincides with the border of $W' = W$ in G. So $W' = W$ is also a standard set in G with the same border, a contradiction to (5).

Case 2: $W \subseteq B$. Let $W'_B = W - \{x,y\} + x*y$. If $\{x,y\} \subseteq W$, then by Fact 15.8 $\rho_{k,\check{G}(x,y)}(W'_B) = \rho_{k,G}(W) - (k+1)(k-2) = 0$, which contradicts Claim 16.

Suppose now that $\{x,y\} \cap W = \emptyset$. Then $W \subset V(\check{G}(x,y))$. By the minimality of G, W contains a standard set W' in $\check{G}(x,y)$, and W' does not contain $x*y$. As in Case 1, by Fact 15.8, W' has the same potential in G as in $\check{G}$, and by Fact 15.7, W' has the same border in G as in $\check{G}$. So W' is also a standard set in G with the same border, a contradiction to (5). Thus, by the symmetry between x and y we may assume $\{x,y\} \cap W = \{x\}$. Let i denote the number of neighbors of y in W.

Case 2A: $i = 0$. Then $\rho_{k,\check{G}(x,y)}(W'_B) = \rho_{k,G}(W) = (k+1)(k-2)$, and by the minimality of G, W'_B contains a standard set W' in $\check{G}(x,y)$. If $x*y \notin W'$, then W' is standard in G exactly as in the previous paragraph. So assume $x*y \in W'$. By the case, y has no neighbors in W', but it does have a neighbor in B. This means, $x*y$ is a border vertex of W' in $\check{G}(x,y)$. But then the set $W'' := W' - x*y + x$ is a standard set in G with the border $W''_* = W'_* - x*y + x$, a contradiction to (5).

Case 2B: $i \geq 1$. By Fact 15.5, we can calculate the potential of W'_B in $\check{G}(x,y)$ exactly:

$$\rho_{k,\check{G}(x,y)}(W'_B) = \rho_{k,G}(W) - 2i(k-1) = k(k-3) - (i-1)2(k-1).$$

By Claim 16 and the definition of Ore-graphs, this yields that $W'_B = V(\check{G}(x,y))$ and $i = 1$. It follows that $W = B - y$, and y has exactly one neighbor, say z in W. But now we are exactly in the situation described by Fact 13 (as B is a k-quasi-xy-edge by Fact 15.2), and so W is a k-quasi-xz-vertex, and therefore a standard set, a contradiction to (5).

Case 3: $W - A \neq \emptyset$ and $W - B \neq \emptyset$. Since $\{x,y\}$ separates $A - \{x,y\}$ from $B - \{x,y\}$ in G, the set $\{x,y\} \cap W$ is a separating set in $G[W]$. Thus by (6), $\{x,y\} \subseteq W$.

Case 3A: $W_A = A$ (equivalently, $A \subseteq W$). We are done, since A is standard by Fact 15.1, Fact 15.3, and the definition of a standard set.

Case 3B: $W_B = B$ (equivalently, $B \subseteq W$). Since $\rho_{k,G}(B) = 2(k^2 - 2k - 1)$ by Fact 15.4 and x, y have no common neighbors in B by 15.5, we have

$$\begin{aligned} \rho_{k,\widetilde{G}(x,y)}(W_A) &= \rho_{k,G}(W) - \rho_{k,G}(B) + \rho_{k,\widetilde{G}(x,y)}(\{x,y\}) \\ &= (k+1)(k-2) - 2(k^2-2k-1) + 2(k+1)(k-2) - 2(k-1) \\ &= (k+1)(k-2). \end{aligned} \tag{8}$$

So for $S = W_A$ the three conditions (a), (b), (c) that imply (7) hold, since (8) implies (a), and statements (b) and (c) are about G (and hence still hold). Thus by (7), W_A contains a standard set W' in $\widetilde{G}(x,y)$. If $|W' \cap \{x,y\}| \le 1$, then W' is a standard set in G with the same border using the same argument as in Case 1. If $\{x,y\} \subset W'$, then replacing in (8) W_A and W with W' and $W' \cup B$ respectively, we get $\rho_{k,G}(W' \cup B) = \rho_{k,\widetilde{G}(x,y)}(W') = (k+1)(k-2)$.

We claim that $W' \cup B$ has the same border in G as W' does in $\widetilde{G}$. By Fact 15.6, the only possible new elements of the border of W' in $W' \cap A$ are x or y. But the only new neighbors of x or y are in B, and $B \subset W' \cup B$ so neither x nor y can be a *new* element of the border. And the neighbors of each $v \in B - x - y$ are in B, so none of the vertices in $B - x - y$ is in the border. This means that $W' \cup B \subseteq W$ is a standard set in G.

Case 3C: $W_A \neq A$ and $W_B \neq B$. Similarly to Case 2, let $W'_B = W_B - \{x,y\} + x * y$. By Claim 16 and since each of $\check{G}$ and $\widetilde{G}$ is k-Ore, $\rho_{k,\widetilde{G}}(W_A) \ge (k+1)(k-2)$ and $\rho_{k,\check{G}}(W'_B) \ge (k+1)(k-2)$. Recall that $\{x,y\} \subset W$ by (6), so $\{x,y\} = W_A \cap W_B$, and so $\rho_{k,\widetilde{G}}(W_A) = \rho_{k,G}(W_A) - 2(k-1)$ and by Fact 15.8, $\rho_{k,\check{G}}(W'_B) = \rho_{k,G}(W_B) - (k+1)(k-2)$. Therefore

$$\begin{aligned} &\rho_{k,G}(W_A) + \rho_{k,G}(W_B) \\ &\ge (k+1)(k-2) + 2(k-1) + (k+1)(k-2) + (k+1)(k-2) \\ &= 3(k+1)(k-2) + 2(k-1). \end{aligned}$$

But since $W_A \cap W_B = \{x,y\}$ and $xy \notin E(G)$, by Fact 11.1,

$$\rho_{k,G}(W_A) + \rho_{k,G}(W_B) = \rho_{k,G}(W) + \rho_{k,G}(\{x,y\}) = 3(k+1)(k-2),$$

a contradiction. ∎

Now we will prove two statements on colorings and structure of subgraphs not containing standard sets of k-Ore graphs.

Lemma 19. *Let G be a k-Ore graph. Let uv be an edge in G such that*

$$\rho_{k,G-uv}(W) > (k+1)(k-2) \quad \text{for every } W \subseteq V(G-uv) \text{ with } 2 \le |W| \le |V(G)| - 1. \tag{9}$$

Then for each $w \in V(G) - u - v$, there is a $(k-1)$-coloring ϕ_w of $G - uv$ such that $\phi_w(w) \neq \phi_w(u) = \phi_w(v)$.

Proof. We use induction on $|V(G)|$. For $G = K_k$, the statement is evident. Otherwise, let $x, y, A, B, \widetilde{G}(x,y)$ and $\check{G}(x,y)$ be as in Fact 14. By Fact 15, $\rho_{k,G}(A) = (k+1)(k-2)$, and thus by (9), $uv \in E(G[A])$.

Case A: $w \in A$. By the induction assumption, there exists a $(k-1)$-coloring ϕ'_w of $\widetilde{G}(x,y)-uv$ such that $\phi'_w(w) \neq \phi'_w(u) = \phi'_w(v)$. Since $\phi'_w(x) \neq \phi'_w(y)$ and B is a quasi-xy-edge, this coloring extends to a $(k-1)$-coloring of the whole $G-uv$.

Case B: $w \in B-x-y$. Let ϕ' be any $(k-1)$-coloring of $\widetilde{G}(x,y)-uv$. By Fact 14(iv), $xy \notin E(G)$, so $uv \neq xy$. By the definition of $\widetilde{G}$, $xy \in E(\widetilde{G})$, so $\phi'(x) \neq \phi'(y)$. Since $\widetilde{G}(x,y)$ is k-critical, $\phi'(u) = \phi'(v)$.

Case B1: $\phi'(u) = \phi'(x)$. Let $G_0 = G[B]+xw$ if $xw \notin E(G)$ and $G_0 = G[B]$ otherwise. Then for each $W \subseteq V(G_0)$,

$$\begin{aligned} \rho_{k,G_0}(W) &= \rho_{k,G}(W) \text{ if } \{x,w\} \not\subseteq W \text{ and} \\ \rho_{k,G_0}(W) &\geq \rho_{k,G}(W) - 2(k-1) \text{ if } \{x,w\} \subseteq W. \end{aligned} \tag{10}$$

Since $\{u,v\} \not\subseteq V(G_0)$, by (9), $\rho_{k,G}(W) > (k+1)(k-2)$ for each $W \subseteq V(G_0)$ with $|W|>1$. This together with (10) imply that

$$\rho_{k,G_0}(W) \geq \rho_{k,G}(W) - 2(k-1) > (k+1)(k-2) - 2(k-1) = k(k-3)$$

for every $W \subseteq V(G_0)$ with $|W|>1$. If $|W|=1$, then $\rho_{k,G_0}(W) = (k+1)(k-2) > k(k-3)$, and so $\rho_{k,G_0}(W) > k(k-3)$ for all nonempty $W \subseteq V(G_0)$. By the second part of Corollary 9, this implies that G_0 has a $(k-1)$-coloring ϕ''. Since $G[B] \subseteq G_0[B]$, coloring ϕ'' is also a coloring of quasi-xy-edge $G[B]$, which yields $\phi''(x) \neq \phi''(y)$. By Fact 14(i) and because $\phi''(x) \neq \phi''(y)$ and $\phi'(x) \neq \phi'(y)$, we can rename the colors in ϕ'' so that $\phi''(x) = \phi'(x)$ and $\phi''(y) = \phi'(y)$, and obtain a $(k-1)$-coloring $\phi = \phi'|_A \cup \phi''|_B$. By construction, $\phi(u) = \phi(x) \neq \phi(w)$.

Case B2: $\phi'(u) \notin \{\phi'(x), \phi'(y)\}$ and $k \geq 5$. Since B is a quasi-edge, $G[B]$ has a $(k-1)$-coloring ϕ'' of $G[B]$ such that $\phi''(x) = \phi'(x)$ and $\phi''(y) = \phi'(y)$. If $\phi''(w) \in \{\phi''(x), \phi''(y)\}$, then by the assumption of the case, $\phi = \phi'|_A \cup \phi''|_B$ is a $(k-1)$-coloring we are looking for. Otherwise, since $k-1 \geq 4$, we can rename the colors of ϕ'' distinct from the colors of x and y so that $\phi''(w) \neq \phi'(u)$ and again take $\phi = \phi'|_A \cup \phi''|_B$.

Case B3: $\phi'(u) \notin \{\phi'(x), \phi'(y)\}$ and $k=4$. Let G_0 be obtained from $G[B]$ by adding a new vertex z adjacent to x, y and w. Suppose first that G_0 has a 3-coloring ϕ''. Since $G[B] \subset G_0$ and B is a quasi-xy-edge, $\phi''(x) \neq \phi''(y)$. So the color of z is distinct from $\phi''(x)$ and $\phi''(y)$, and thus because there are only 3 colors, $\phi''(w) \in \{\phi''(x), \phi''(y)\}$. In this case by renaming the colors in ϕ'' so that $\phi''(x) = \phi'(x)$ and $\phi''(y) = \phi'(y)$, we get a required coloring of G. Now suppose that G_0 has no 3-coloring. Then G_0 contains a 4-critical subgraph G_1. Since G_1 is not a subgraph of G, it follows that $z \in V(G_1)$. Since G_1 is

4-critical, $\delta(G_1)\ge 4-1=3$, and so $\{x,y,w\}\subset V(G_1)$. Let $W=V(G_1)$. Since $\rho_{4,G_0}(W)\le 4$ by Corollary 9, we have $\rho_{4,G}(W-z)=\rho_{4,G}(W)-10+3(6)\le 12$. So Fact 11.1 (because $G[A\cap W]=G[\{x,y\}]\cong 2K_1$) implies that

$$\rho_{4,G}(A\cup W-z)\le \rho_{4,G}(A)+\rho_{4,G}(W-z)-2\rho_4(K_1)\le 10+12-20=2.$$

By Claim 16, this yields that $A\cup W-z$ either is empty or is $V(G)$. But $|A\cup W-z|\ge 3$ because $\{x,y,w\}\subset W-z$. Also $A\cup W-z\neq V(G)$, since G is 4-Ore, and the vertex set of each 4-Ore graph has potential $k(k-3)=4$, since it is 4-extremal. ∎

Claim 20. *Let G be a k-Ore graph. Let u be a vertex in G such that*

(11) $\quad \rho_{k,G}(W)>(k+1)(k-2)$ *for every* $W\subseteq V(G)-u$ *with* $|W|\ge 2$.

Then there exists a $(k-1)$-clique $S\subseteq V(G)-u$ such that $d_G(v)=k-1$ for all $v\in S$ and $N(S)-S$ is an independent set.

Proof. We use induction on $|V(G)|$. For $G=K_k$, the statement is evident. Otherwise, let $x,y,A,B,\widetilde{G}(x,y)$ and $\check{G}(x,y)$ be as in Fact 14. Then $\rho_{k,G}(A)=(k+1)(k-2)$, and so $u\in A$.

If there exists a $W\subseteq V(\check{G}(x,y))$ such that $|W|\ge 2$ and $\rho_{k,\check{G}(x,y)}(W)\le (k+1)(k-2)$, then by (11), $x*y\in W$. So by induction, $\check{G}$ has a set $S\subseteq V(\check{G}(x,y))-x*y$ such that $\check{G}(x,y)[S]\cong K_{k-1}$, $d_{\check{G}(x,y)}(v)=k-1$ for all $v\in S$, and $N(S)-S$ is an independent set in $\check{G}(x,y)$. Recall that $\check{G}-x*y$ is a subgraph of G. So since $u\in A$, we have $S\subseteq V(G)-u$, $G[S]\cong K_{k-1}$, and $N_G(S)-S$ is an independent set in G. By Fact 15.7, $d_G(v)=k-1$ for all $v\in S$. ∎

3. Basic properties of minimal counter-examples

The *closed neighborhood* of a vertex u in a graph H is $N_H[u]=N_H(u)\cup\{u\}$. We will use the following partial order on the set of graphs. A graph H is *smaller than* a graph G, if either

(S1) $|V(G)|>|V(H)|$, or
(S2) $|V(G)|=|V(H)|$ and $|E(G)|>|E(H)|$, or
(S3) $|V(G)|=|V(H)|$, $|E(G)|=|E(H)|$ and G has fewer pairs of adjacent vertices with the same closed neighborhood.

Note that if H is a subgraph of G, then H is smaller than G. Let $k\ge 4$ and J be a minimal with respect to relation "smaller" counter-example to Theorem 10: J is a k-critical graph with $\rho_k(V(J)) > y_k$ that is not k-Ore. Since y_k and the values of ρ_k are always even, the restriction $\rho_k(V(J)) > y_k$ is equivalent to

$$\rho_k(V(J)) \ge y_k + 2. \tag{12}$$

Let $n := |V(J)|$. In this section, we derive basic properties of J and its colorings.

Claim 21. *J is 3-connected.*

Proof. Suppose that J has a separating set $\{x,y\}$ and sets $A\subset V(J)$ and $B\subset V(J)$ such that $A\cap B=\{x,y\}$, $A\cup B=V(J)$, and no edge of J connects $A-x-y$ with $B-x-y$. By Fact 12 and the symmetry between A and B, we may assume that A is a k-quasi-xy-vertex and B is a k-quasi-xy-edge. It follows that the graph $\widetilde{J}$ obtained from $J[A]$ by inserting edge xy and the graph $\check{J}$ obtained from $J[B]$ by gluing x with y are k-critical. Then

$$\begin{aligned}\rho_k(V(J)) &\le (\rho_k(V(\widetilde{J})) + 2(k-1)) + (\rho_k(V(\check{J})) \\ &\qquad + (k+1)(k-2)) - 2\cdot(k+1)(k-2) \\ &= \rho_k(V(\widetilde{J})) + \rho_k(V(\check{J})) - k(k-3).\end{aligned} \tag{13}$$

By assumption, $y_k < \rho_k(V(J))$. By Corollary 9, $\rho_{k,\widetilde{J}}(V(\widetilde{J})) \le k(k-3)$ and $\rho_k(V(\check{J})) \le k(k-3)$. Moreover, if $\widetilde{J}$ (respectively, $\check{J}$) is not a k-Ore graph, then by the minimality of J, the potential of its vertex set is at most y_k. If at least one of $\widetilde{J}$ or $\check{J}$ is not k-Ore, then we get a contradiction with (13). If both are k-Ore, then J is k-Ore, which contradicts the definition of J. ∎

Fact 22. *By the definition of ρ_k and the assumption $\rho_k(V(J)) > y_k$, for each $v\in V(J)$,*

$$\rho_k(V(J)-v) = \rho_k(V(J)) - (k+1)(k-2) + 2(k-1)d(v) >$$

- $y_k + k^2 - 3k + 4$, *if* $d(v) = k-1$,
- $y_k + k^2 - k + 2$, *if* $d(v) = k$,
- $y_k + k^2 + k$, *if* $d(v) \ge k+1$.

Because $y_k \ge k^2 - 5k + 2$, we see that $\rho_k(V(J)-v)$ is also more than

- $2k^2 - 8k + 6 = 2(k-3)(k-1)$, *if* $d(v) = k-1$,
- $2k^2 - 6k + 4 = 2(k-2)(k-1)$, *if* $d(v) = k$,

- $2k^2-4k+2=2(k-1)^2$, *if* $d(v)\geq k+1$.

Now we define graph $Y(J,R,\phi,X)$. The idea of $Y(J,R,\phi,X)$ is that it is often smaller than J, and every $(k-1)$-coloring of it extends to a $(k-1)$-coloring of J.

Definition 23. For a graph G, a set $R\subset V(G)$ and a $(k-1)$-coloring $\phi\colon R\to[k-1]$ of $G[R]$, the graph $Y(G,R,\phi,X)$ is constructed as follows. Let R_* be the border of R, i.e., $R_*=\{v\in R\colon N(v)-R\neq\emptyset\}$. Let t be the number of colors used by ϕ on R_*. We may renumber the colors so that the colors used by ϕ on R_* are $1,\dots,t$. First, for $i=1,\dots,t$, let R'_i denote the set of vertices in $V(G)-R$ adjacent in G to at least one vertex $v\in R$ with $\phi(v)=i$. Now, let $Y(G,R,\phi,X)$ be obtained from $G-R$ by adding a set $X=\{x_1,\dots,x_t\}$ of new vertices such that $N(x_i)=R'_i\cup(\{x_1,\dots,x_t\}-x_i)$ for $i=1,\dots,t$.

Informally, the definition can be rephrased as follows: For a given $R\subset V(G)$ and a $(k-1)$-coloring ϕ of $G[R]$, we glue each color class of $\phi(G[R])$ into a single vertex, then add all possible edges between the new vertices (corresponding to the color classes) and then delete those that have no neighbors outside of R. Graph $Y(G,R,\phi,X)$ will be a helpful gadget for deriving properties of G, since it inherits some structure from G.

First we will prove some properties of $Y(J,R,\phi,X)$.

Claim 24. *Suppose* $R\subset V(J)$ *and* ϕ *is a* $(k-1)$*-coloring of* $J[R]$. *Then* $\chi(Y(J,R,\phi,X))\geq k$.

Proof. Let $Y=Y(J,R,\phi,X)$. Suppose Y has a $(k-1)$-coloring ϕ'. By the construction of Y, the colors of all x_i in ϕ' are distinct. We can change the names of the colors so that $\phi'(x_i)=i$ for $1\leq i\leq t$, where t is given in Definition 23. Again by the construction of Y, $\phi'(u)\neq i$ for each vertex $u\in R'_i$. Therefore $\phi|_R\cup\phi'|_{V(J)-R}$ is a proper coloring of J, a contradiction. ∎

The next claim is a submodularity-type equation that is a direct extension of Fact 11.1.

Claim 25. *Let* $R\subset V(J)$, ϕ *be a* $(k-1)$*-coloring of* $J[R]$ *and* $Y=Y(J,R,\phi,X)$. *Let* $W\subseteq V(Y)$. *If* $W\cap X=\{x_{i_1},\dots,x_{i_q}\}$, *then let* $R|_W$ *denote the set of vertices* $v\in R_*$ *such that* $\phi(v)\in\{i_1,\dots,i_q\}$. *Then*

$$\begin{aligned}\rho_{k,J}(W-X+R)&=\rho_{k,Y}(W)-\rho_{k,Y}(W\cap X)\\&\quad+\rho_{k,J}(R)-2(k-1)|E_J(W-X,R-R|_W)|.\end{aligned}\tag{14}$$

Proof. Since $\rho_{k,J}(U)$ is a linear combination of the numbers of vertices and edges in $J[U]$, it is enough to check that the weight of every vertex and edge of $J[W-X+R]$ is accounted exactly once in the RHS of (14) and the weight of every other vertex or edge either does not appear at all or appears once with plus and once with minus. In particular, the weight of every vertex and edge of $Y[W\cap X]$ appears once with plus and once with minus. ∎

By Corollary 9 and Claim 24, $Y(J,R,\phi,X)$ contains a vertex set with potential at most $k(k-3)$. In some instances this will not be enough for our purposes, and we will want $Y(J,R,\phi,X)$ to contain a vertex set with potential at most y_k. The next claim helps us with this.

Claim 26. *Let $R\subset V(J)$, ϕ be a $(k$–$1)$-coloring of $J[R]$ and $Y=Y(J,R,\phi,X)$. Then Y contains a k-critical subgraph Y', and so $\rho_{k,Y}(V(Y'))\le k(k-3)$. Furthermore, if $|R|\ge k$, then Y is smaller than J and*

(a) either Y' is an induced k-Ore subgraph, or
(b) $\rho_{k,Y}(V(Y'))\le y_k$.

Moreover, $V(Y')\cap X\ne\emptyset$.

Proof. By Claim 24, Y has a k-critical subgraph Y'. The bound on the potential of $V(Y')$ follows from Corollary 9. In order to prove the "Furthermore" part, observe that if $|R|\ge k$, then Y is smaller than J by Rule (S1) in the definition of "smaller", since ϕ uses at most $k-1<|R|$ colors on R. Because Y' is a subgraph of Y, Y' is smaller or equal to Y, and so by transitivity is smaller than J. Thus, by the minimality of J, either Y' is k-Ore or $V(Y')$ has potential at most y_k. If Y' is an induced subgraph and k-Ore, then (a) holds. If Y' is not induced, then by Fact 11.10, (b) holds. If Y' is not k-Ore, then (b) holds by the minimality of J.

Since $\chi(Y')>\chi(J[R])$, Y' is not a subgraph of J. So, $V(Y')\cap X\ne\emptyset$. ∎

Now we will use $Y(J,R,\phi,X)$ to prove lower bounds on potentials of nontrivial sets.

Claim 27. *If $\emptyset\ne R\subsetneq V(J)$, then $\rho_{k,J}(R)\ge\rho_k(V(J))+2(k-1)>y_k+2(k-1)$.*

Proof. Let R be a nonempty proper subset of $V(J)$ with the smallest potential. Since J is k-critical, $J[R]$ has a proper coloring $\phi\colon R\to[k-1]$. Let $Y=Y(J,R,\phi,X)$. By Claim 26, Y contains a subset S with potential at most $k(k-3)$ and $S\cap X\ne\emptyset$. Let $Z=S-X+R$. Because $|X|\le k-1$, by Fact 11.7, each non-empty subset of X has potential at least $(k+1)(k-2)$. So by (14),

$$\begin{aligned}\rho_{k,J}(Z) &\le \rho_{k,Y}(S)-\rho_{k,Y}(S\cap X)+\rho_{k,J}(R)\\ &\le k(k-3)-(k+1)(k-2)+\rho_{k,J}(R)=\rho_{k,J}(R)-2(k-1).\end{aligned}\tag{15}$$

Since $Z \supset R$, it is nonempty. So, by the minimality of the potential of R, we have $Z = V(J)$.

The final statement comes from our assumption that $\rho_k(V(J)) > y_k$. ∎

The following fact implies that J contains no quasi-vertex.

Claim 28. *For each $R \subsetneq V(J)$ with $|R| \geq 2$ and any distinct $x, y \in R$, the graph $J[R] + xy$ is $(k-1)$-colorable.*

Proof. Let R be a smallest subset of vertices of J such that $2 \leq |R| < n$ and for some distinct $x, y \in R$, the graph $H = J[R] + xy$ is not $(k-1)$-colorable. Since J is k-critical, $xy \notin E(J)$.

Let H' be a k-critical subgraph of H. By the minimality of R, $V(H') = R$. By Claim 27, $\rho_{k,H'}(R) = -(2k-2) + \rho_{k,J}(R) > y_k$. Because $|R| < n$, by Rule (S1), H' is smaller than J. So by the minimality of J and because $\rho_{k,H'}(R) > y_k$, H' is k-Ore and $\rho_{k,H'}(R) = k(k-3)$. If there is an edge $e \in E(H) - E(H')$, then

$$\rho_{k,H}(R) \leq \rho_{k,H'}(R) - 2(k-1) = k(k-3) - 2(k-1) \leq y_k,$$

contradicting Claim 27. Hence $H = H'$. Thus H is k-Ore and

$$\rho_{k,J}(R) = \rho_{k,H}(R) + 2(k-1) = k(k-3) + 2(k-1) = (k+1)(k-2). \tag{16}$$

By Claim 21, $|R_*| \geq 3$. We want to prove that

(17) $J[R]$ has a $(k-1)$-coloring ψ such that R_* is not monochromatic.

Case 1: $\{x, y\} \subset R_*$. Since $|R_*| \geq 3$, we may choose $w \in R_* - x - y$. If there exists a subset $R' \subsetneq R$ with $|R'| \geq 2$ such that $\{x, y\} \not\subset R'$ and $\rho_k(R') = (k+1)(k-2)$, then by Lemma 18, H contains a standard set $A \subseteq R'$. But then there exists a pair of vertices $\{a, b\} \subset A \subseteq R' \subsetneq R$ such that $J[A] + ab$ is not $(k-1)$-colorable, which contradicts the minimality of R. Otherwise, by Lemma 19, there is a $(k-1)$-coloring ϕ_w of $H - xy$ such that $\phi_w(w) \neq \phi_w(x) = \phi_w(y)$. Then for $\psi = \phi_w$, (17) holds.

Case 2: $\{x, y\} \not\subset R_*$. Let u, v be any vertices in R_*. If $uv \in E(J)$, then (17) is immediately true. Otherwise, let $H_0 = J[R] + uv$. If H_0 has a $(k-1)$-coloring, then (17) holds. If not, then by the minimality of R, exactly as above, H_0 is a k-Ore graph. So, we have Case 1. This proves (17).

Let ψ satisfy (17). Let $Y = Y(J, R, \psi, X)$. By Claim 26, Y contains a vertex set W such that $\rho_{k,Y}(W) \leq k(k-3)$ and $W \cap X \neq \emptyset$. Recall that X induces a complete graph and that by Fact 11.7 the subgraph of K_{k-1} with the smallest potential is K_1 with $\rho_k(V(K_1)) = (k+1)(k-2)$ and the subgraph with the second smallest potential is K_{k-1} with $\rho_k(V(K_{k-1}))=2(k–2)(k–1)$. This together with (16) and the choice of W yields

$$\begin{aligned} \rho_{k,J}(W-X+R) &\leq \rho_{k,Y}(W)-\rho_{k,Y}(X\cap W) \\ +\rho_{k,J}(R) &\leq k(k-3)-(k+1)(k-2)+(k+1)(k-2)=k(k-3). \end{aligned} \tag{18}$$

Since $W-X+R\supset R$, we have $|W-X+R|\geq 2$. By Fact 11.8, $y_k+(2k-2)\geq k(k–3)$. This together with Claim 27 yields $W–X+R=V(J)$. If $|W\cap X|\geq 2$, then we get the stronger bound $\rho_k(X\cap W)\geq 2(k-1)(k-2)$, and so the inequality in (18) strengthens to

$$\rho_{k,J}(W-X+R)\leq k(k-3)-2(k-1)(k-2)+(k-2)(k+1)=2k-6\leq y_k,$$

a contradiction. Thus $|X\cap W|=1$. Because R_* is not monochromatic and $|X\cap W|=1$, there is a vertex $z\in R_*-W$. Then by (14), instead of (18) we have

$$\begin{aligned} &\rho_{k,J}(W-X+R) \\ \leq k(k-3)-(k+1)(k-2)&+(k+1)(k-2)-2k+2=k^2-5k+2\leq y_k, \end{aligned}$$

a contradiction. ∎

Claim 29. *Let X be a $(k-1)$-clique in J, $u,v\in X$, $N(u)-X=\{a\}$, and $N(v)-X=\{b\}$. Then $a=b$.*

Proof. Assume $a\neq b$. Let $J'=J-u-v+ab$ if $ab\notin E(J)$ and $J'=J-u-v$ otherwise. By Claim 28, J' has a $(k–1)$-coloring ϕ. Because $d(u)=d(v)=k–1$, each of the sets $C_a=\{1,\dots,k-1\}-\cup_{w\in N(u)-v}\phi(w)$ and $C_b=\{1,\dots,k-1\}-\cup_{w\in N(v)-u}\phi(w)$ is nonempty. Since $\phi(a)\neq\phi(b)$ and $(N(u)-a)=(N(v)-b)$, the sets C_a and C_b are different. Therefore ϕ can be extended to u and v. But then we get a $(k-1)$-coloring of J, a contradiction. ∎

Claim 30. *J does not contain K_k-e.*

Proof. Suppose $J[R]=K_k-e$. The only k-critical graph on k vertices is K_k, which is k-Ore. By assumption J is not k-Ore, so $R\neq V(J)$, but adding the missing edge to $J[R]$ creates a k-chromatic graph on R, a contradiction to Claim 28. ∎

4. Clusters and sets with small potential

Recall that in Section 3 we defined a relation "smaller," and J is a "smallest" counterexample to Theorem 10: it is a k-critical graph with $\rho_k(V(J)) > y_k$ and is not k-Ore.

Definition 31. For $S \subseteq V(J)$, an S-*cluster* is an inclusion maximal set $R \subseteq S$ such that for every $x \in R$, $d(x) = k-1$ and for every $x,y \in R$, $N[x] = N[y]$. A *cluster* is a $V(J)$-cluster.

Each cluster is a clique of vertices with degree k–1. In this section, results on clusters will help us to derive the main lower bound on the potentials of nontrivial vertex sets, Lemma 38, which in turn will help us to prove stronger results on the structure of clusters in J.

Having the same closed neighborhood is an equivalence relation, and so the set of clusters is a partition of the set of the vertices with degree $k-1$. Thus the following fact holds.

Fact 32. *Every vertex with degree $k-1$ is in a unique cluster.*

Furthermore, if the only S-cluster is the empty set, then every vertex in S has degree at least k. By definition, if a cluster T is contained in a vertex set S, then T is also an S-cluster.

Claim 33. *Every cluster T satisfies $|T| \leq k-3$. Furthermore, for every $(k-1)$-clique X in J, (i) there is a unique X-cluster T (possibly $T = \emptyset$), and (ii) every non-empty X-cluster is a cluster (in other words, every cluster is either contained in X or disjoint from X). Each $(k-1)$-clique in J contains at least 2 vertices of degree at least k.*

Proof. If T is a cluster with $|T| \geq k$–2, then $T \cup N(T) \supseteq K_k$–$e$, a contradiction to Claim 30.

Let X be a $(k-1)$-clique in J. Two distinct X-clusters would contradict Claim 29. If T is a non-empty X-cluster contained in a larger cluster T', then each $v \in T' - X$ has to be adjacent to each vertex in X, and so J contains clique $X \cup T'$ of size at least k, a contradiction.

The final statement is proven as follows: by Fact 32 only vertices in clusters have degree $k-1$, by (i) X contains at most one cluster which, if exists, has at most $k-3$ of the $k-1$ vertices in X by the first part of this claim. ∎

Claim 34. *Let $R \subset V(J)$, let ϕ be a $(k-1)$-coloring of $J[R]$, and let $Y = Y(J, R, \phi, X)$ be as in Definition 23. If Y is k-Ore, then $Y \cong K_k$.*

Proof. Suppose that Y is a k-Ore graph distinct from K_k. Let a separating set $\{x,y\}$, vertex subsets $A = A(Y,x,y)$ and $B = B(Y,x,y)$, and graphs $\widetilde{Y}(x,y)$ and $\check{Y}(x,y)$ be as in Fact 14. Since $Y[X]$ is a clique and $E_Y(A-x-y, B-x-y)=\emptyset$, either $X \subseteq A$ or $X \subseteq B$. Since $xy \notin E(Y)$ we may assume that either $X \subset A-y$ or $X \subset B-y$. By construction, $Y-J=X$, so this implies that $B-x$ or $A-x$, respectively, is a subgraph of J. We will show that this is impossible.

Suppose first that $X \subseteq A-y$. The graph $\check{Y}-x*y$ is a subgraph of J, namely, it is $J[B-x-y]$, and by Fact 15.7

$$d_{\check{Y}}(v) = d_J(v) \text{ for every } v \in B-x-y. \tag{19}$$

If $\check{Y}-x*y$ has a vertex subset S with $|S|\geq 2$ of potential at most $(k+1)(k-2)$, then by Lemma 18, S contains a standard set S'. But each standard set S' has two vertices u and w such that $Y[S']+uw$ is not $(k-1)$-colorable. This contradicts Claim 28. Thus $\rho_{k,\check{Y}}(S)>(k+1)(k-2)$ for every $S\subseteq V(\check{Y})-x*y$ with $|S|\geq 2$. Then by Claim 20, there exists an $S\subseteq V(\check{Y})-x*y=B-x-y$ such that $\check{Y}[S]\cong K_{k-1}$, and $d_{\check{Y}}(v)=k-1$ for all $v\in S$. By (19), this contradicts the last part of Claim 33.

Now suppose that $X\subseteq B-y$. The graph $\widetilde{Y}-x$ is a subgraph of J, namely, it is $J[A-x]$, and similarly to (19), by Fact 15.6

$$d_{\widetilde{Y}}(v) = d_J(v) \text{ for every } v \in A-x-y. \tag{20}$$

As above, $\rho_{k,\widetilde{Y}}(S)>(k+1)(k-2)$ for every $S\subseteq V(\widetilde{Y})-x$ with $|S|\geq 2$. So again by Claim 20, there exists an $S'\subseteq V(\widetilde{Y})-x=A-x$ such that $\widetilde{Y}[S']\cong K_{k-1}$, and $d_{\widetilde{Y}}(v)=k-1$ for all $v\in S'$. But $|S'-y|\geq k-2$, which together with (20) contradicts Claim 33. ∎

Claim 35. *For every partition (A,B) of $V(J)$ with $2 \leq |A| \leq n-2$, $|E_J(A,B)|\geq k$.*

Proof. Let A_* (respectively, B_*) be the set of vertices in A (respectively, B) that have neighbors in B (respectively, A). Since J is 3-connected, $|A_*|\geq 3$ and $|B_*|\geq 3$. So by Claim 28, $J[A]$ has a $(k-1)$-coloring ϕ_A such that A_* is not monochromatic, and $J[B]$ has a $(k-1)$-coloring ϕ_B such that B_* is not monochromatic. But Gallai and Toft (see [32, p. 157]) independently proved that if $|E_J(A,B)|\leq k-1$, then either A_* is monochromatic in every $(k-1)$-coloring of $J[A]$ or B_* is monochromatic in every $(k-1)$-coloring of $J[B]$. So, $|E_J(A,B)|\geq k$. ∎

Sometimes below, our goal will be to extend to J a coloring ϕ of $J[R]$ for some R and ϕ. Recall that $Y(J,R,\phi,X)$ is obtained from J by replacing the vertices of R with a clique whose vertices are the color classes of ϕ with at least one element in the border of R (which we called R_*). One of the ways we will control ϕ is to add edge(s) to R before we generate a $(k-1)$-coloring ϕ using Claim 28 and a lemma below. Our next lemma describes how edges can be placed in R so that no color class of ϕ is too large. The proof of this lemma will use the following old result of Hakimi.

Theorem 36 (Hakimi [20]). *Let $(w_1, \ldots, w_s)$ be a list of nonnegative integers with $w_1 \geq \ldots \geq w_s$. Then there is a loopless multigraph F with vertex set $\{u_1, \ldots, u_s\}$ such that $d_F(u_j) = w_j$ for all $j = 1, \ldots, s$ if and only if $z = w_1 + \ldots + w_s$ is even and $w_1 \leq w_2 + \ldots + w_s$.*

For technical reasons, in one specific case of the lemma below we will allow for a hyperedge of size 3. Recall that an *independent set* in a hypergraph is a set that contains no edge.

Lemma 37. *Let $i' \geq 1$ and $s \geq 2$ be integers. Let $R_* = \{u_1, \ldots, u_s\}$ be a vertex set. Then for each $z \geq 2i'$ and any integral positive weight function $w \colon R_* \to \{1, 2, \ldots\}$ such that $w(u_1) + \ldots + w(u_s) = z$ and $w(u_1) \geq w(u_2) \geq \ldots \geq w(u_s)$, there exists a graph H with $V(H) = R_*$ and $|E(H)| \leq i'$ such that for each $1 \leq j \leq s$, $d_H(u_j) \leq w(u_j)$, and for every independent set M in H with $|M| \geq 2$,*

$$\sum_{u \in R_* - M} w(u) \geq i'. \tag{21}$$

Moreover, if $s \geq 3$, $i' \geq 1$, and $z > 2i'$, then at least one of the three stronger statements below holds:

(i) such H with Property (21) could be chosen as a graph with at most $i' - 1$ edges, or
(ii) such H with Property (21) could be chosen as a hypergraph instead of a graph with at most $i' - 1$ graph edges and one edge of size 3, or
(iii) the weight arrangement is i'-special, which means that $s = i' + 1$ and $w(u_2) = \ldots = w(u_s) = 1$.

Proof. It is easy to check that the main part of the statement, and (ii) of the "moreover" part hold for $i' = 1$. So assume $i' \geq 2$.

Case 1: $w(u_2) + \ldots + w(u_s) \leq i' - 1$. We make $E(H) = \{u_1 u_j \colon 2 \leq j \leq s\}$. If M is any independent set with $|M| \geq 2$, then $u_1 \notin M$ and $w(u_1) \geq 2i' - (i' - 1)$ yielding (21). To prove the "Moreover" part in this case, observe that our H has at most $i' - 1$ edges.

Case 2: $s \geq 2i'+1$. Let the edge set of H consist of the matching $\{u_1u_2, \dots, u_{2i'-1}u_{2i'}\}$. Every independent set misses at least one end of each edge in H, which implies (21). Moreover, if $z \geq 2i'+1$, then we extend edge u_1u_2 to the hyperedge $\{u_1, u_2, u_{2i'+1}\}$. Then Claim (ii) of the "Moreover" part of the lemma holds.

Case 3: $w(u_2)+\ldots+w(u_s) \geq i'$ and $s \leq 2i'$. Since $s \leq 2i'$, there exists an auxiliary integral weight function $w' \colon R_* \to \{1, 2, \dots\}$ such that

$$w'(u_2) + \ldots + w'(u_s) \geq i', w'(u_1) + \ldots + w'(u_s) = 2i' \\ \text{and } w'(u_j) \leq w(u_j) \; \forall\, j = 1, \dots, s. \tag{22}$$

By (22) and Theorem 36, there exists a loopless multigraph H' with vertex set $\{u_1, \dots, u_s\}$ such that $d_{H'}(u_j) = w'(u_j)$ for all j. We obtain a graph H from the multigraph H' by replacing each set of multiple edges with a single edge. Every independent set in H is also independent in H'. For every independent set M in H', each of its i' edges has an end outside of M, so

$$\sum_{u \in R_* - M} w(u) \geq \sum_{u \in R_* - M} w'(u) = \sum_{u \in R_* - M} d_{H'}(u) \geq |E(H')| = i'.$$

This yields (21). Note that in this case, (21) holds for *every* independent set M, even if $|M|=1$.

Now we prove the "Moreover" part of the statement. If H' had any multiple edge, then we satisfy (i) and are done. Suppose, H' is simple. Since $z > 2i'$, $w'(u_\ell) < w(u_\ell)$ for some $1 \leq \ell \leq s$. If $H - u_\ell$ has an edge e, then after enlarging e to $e+u_l$ we still keep (21). This instance satisfies (ii), and we are done. Otherwise u_ℓ is incident to every edge of $H = H'$, and so H is a star with center u_ℓ and $i' \geq 2$ edges. Each such star has only one central vertex, so every other vertex u_j satisfies $w(u_j) = w'(u_j) = d_H(u_j) = 1$. By definition, this means that the weight arrangement is i'-special. So we satisfy (iii) and are done. ∎

Recall that $\rho_{k,K_{k-1}}(V(K_{k-1})) = 2(k-1)(k-2)$. Importantly, this is larger than the potential of a standard set. Our main lower bound on the potentials of nontrivial vertex sets is the following.

Lemma 38. *If $R \subsetneq V(J)$ and $2 \leq |R| \leq n-2$, then $\rho_k(R) \geq 2(k-1)(k-2)$. Moreover, if $\rho_k(R) = 2(k-1)(k-2)$, then $J[R] = K_{k-1}$.*

Proof. Assume that the lemma does not hold. Let i be the smallest integer such that there exists $R \subsetneq V(J)$ with

$$2 \leq |R| \leq n-2,\ J[R] \neq K_{k-1},\ \rho_k(R) \leq 2(k-1)(k-2), \tag{23}$$

and

$$y_k + 2i(k-1) < \rho_k(R) \le y_k + 2(i+1)(k-1). \tag{24}$$

It is important that we are only minimizing i, and not necessarily minimizing $\rho_k(R)$. By Claim 27, $i \ge 1$. In order for both (23) and (24) to hold, since

$$y_k + (k+1)(k-1) \ge k^2 - 5k + 2 + (k+1)(k-1) > 2(k-1)(k-2),$$

$i \le \frac{k}{2}$. By the integrality, $i \le \lfloor \frac{k}{2} \rfloor$. Moreover, if $k=4$, then $y_k = \max\{2\cdot 4 - 6, 4^2 - 5\cdot 4 + 2\} = 2$ and so $y_4 + 4(4-1) = 14 > 12 = 2(4-1)(4-2)$. Thus

$$i \le \left\lfloor \frac{k}{2} \right\rfloor, \text{ moreover, if } k=4, \text{ then } i=1. \tag{25}$$

Let R be a smallest subset of $V(J)$ for which (23) and (24) hold. By Fact 11.7, each vertex set S with $2 \le |S| \le k-1$ has potential at least $2(k-1)(k-2)$, with equality only when S induces a K_{k-1}. So (23) yields $|R| \ge k$. Thus by Claim 26, for any proper $(k-1)$-coloring ϕ of $J[R]$, graph $Y(J,R,\phi,X)$ is smaller than J.

Let $Q = V(J) - R$, and for $u \in R$, let $w(u) = |N(u) \cap Q|$. By Definition 17, $R_* = \{u \in R\colon w(u) \ge 1\}$. Let $R_* = \{u_1, \dots, u_s\}$ and $w(u_1) \ge \dots \ge w(u_s)$. By Claim 35,

$$z := \sum_{i=1}^{s} w(u_i) = |E_J(R, V(J) - R)| \ge k. \tag{26}$$

By Claim 21, $s \ge 3$.

We will consider four cases, and the first is the main one.

Case 1: There is a $(k-1)$-coloring ϕ of $J[R]$ such that for every color class C of ϕ with $|C \cap R_*| \ge 2$ either

$$\textstyle\sum_{u \in R_* - C} w(u) \ge i \tag{27}$$

or

$$\sum_{u \in R_* - C} w(u) = i - 1 \text{ and } \sum_{u \in C} w(u) \le k - 2. \tag{28}$$

Let $Y = Y(J,R,\phi,X)$ be as in Definition 23.

Let Y' be the k-critical subgraph of Y described in Claim 26. Let $W = V(Y')$ and $X' = X \cap W$. Since $|R| \ge k$, by Claim 26, $X' \ne \emptyset$ and one of the following two statements is true: (a) $Y[W]$ contains a k-Ore graph, or (b) $\rho_{k,Y}(W) \le y_k$. Because $X' \ne \emptyset$, by Fact 11.7, $\rho_{k,Y}(X') \ge (k+1)(k-2)$. By

Corollary 9, $\rho_{k,Y}(W) \le \rho_{k,Y'}(W) \le k(k-3)$. By (14) and our bounds on $\rho_{k,Y}(X')$ and $\rho_{k,Y}(W)$,

(29) $\rho_{k,J}(W-X+R) \le \rho_{k,Y}(W)-\rho_{k,Y}(X')+\rho_{k,J}(R) \le \rho_{k,J}(R)-2(k-1).$

Because $|W-X+R|\ge|R|\ge k$ (which implies $J[W-X+R]\ne K_{k-1}$) and the choice of i, $|W-X+R|\ge n-1$. Suppose first that $|W-X+R|=n-1$. By Fact 22, $\rho_{k,J}(W-X+R)>y_k+k^2-3k+4$. By (29), $\rho_{k,J}(R)>y_k+k^2-k+2\ge 2(k-1)(k-2)$, contradicting the choice of R. So

$$W-X+R=V(J). \tag{30}$$

We claim that Y is a k-clique and will prove this in three steps. We will show, in order, that

(31) (A) $|X'|\ge 2$, (B) Y' is an induced k-Ore graph, and (C) $Y'=Y$.

These three steps prove that Y is k-Ore; Claim 34 yields that if Y is k-Ore, then Y is a k-clique.

Suppose $X'=\{x_j\}$. Then $W=V(Y')-X+x_j$. Let $R_j=\{u\in R_*\colon \phi(u)=c_j\}$, where c_j is the jth color class in coloring ϕ. If $|R_j|=1$, then $Y'\cong J[W-x_j\cup R_j]$, which is a subgraph of J. Because $|R|\ge k>1=|R_j|$, Y' is a proper subgraph of J, but k-critical graphs do not have k-chromatic proper subgraphs. Thus $|R_j|\ge 2$. If (28) holds for R_j, then $d_{Y'}(x_j)\le\sum_{u\in R_j} w(u)\le k-2$, but the k-critical graph Y' cannot have vertices of degree less than $k-1$. Otherwise, by (27), at least i edges connect the vertices in R_*-R_j with Q. Those edges are a part of $E_J(W-X,R-R|_W)$ in (14), an edge set we conservatively estimated to be empty in the calculation of (29). Adjusting that calculation to account for our new bound on $E_J(W-X,R-R|_W)$, instead of (29) we get

$$\begin{aligned}&\rho_{k,J}(W-\{x_j\}+R)\\ \le\ & k(k-3)-(k-2)(k+1)+\rho_{k,J}(R)-2i(k-1)\\ =\ & \rho_{k,J}(R)-2(i+1)(k-1).\end{aligned}$$

By (24), we have $\rho_{k,J}(R)-2(i+1)(k-1)\le y_k$, which together with $i\ge 0$, (29), and (30) imply $\rho_{k,J}(V(J))\le y_k$, contradicting our assumption that $\rho_{k,J}(V(J))>y_k$. This proves (A).

If Y' is not an induced k-Ore graph, then by Claim 26, $\rho_{k,Y}(W)\le y_k$. We adjust the calculation of (29) from (14) again: we use the inequality $\rho_{k,Y}(W)\le y_k$ and that part (A) implies $2\le|X'|\le k-1$, so X' has potential at least $2(k-1)(k-2)$ by Fact 11.7. The result is

$$\rho_{k,J}(W-X+R)\le y_k-2(k-1)(k-2)+\rho_{k,J}(R).$$

We apply (30) to the left hand side and (23) to the right hand side to see that $\rho_{k,J}(V(J)) \leq y_k$, which contradicts our choice of J. This proves (B).

By (30) and (B), if $Y' \neq Y$, then $X' \neq X$. If $X' \neq X$, then we again get a stronger bound on $E_J(W-X, R-R|_W)$ in (14) (this time it contains at least one element), and so we may again adjust the bound in (29) by subtracting $2(k-1)$. We augment the application of (14) to (29) in two ways: the extra $-2(k-1)$ described in the previous sentence and that $|X'| \geq 2$ implies $\rho_{k,Y}(X') \geq 2(k-1)(k-2)$ by Fact 11.7; those augmentations and (23) produce

$$\rho_{k,J}(W-X+R) \leq -2(k-1)+k(k-3)-\rho_{k,Y}(X')+\rho_{k,J}(R) \leq k^2-5k+2 \leq y_k,$$

which contradicts our choice of J for the same reason as the proof to part (B). This proves (C); and therefore $Y = Y(J, R, \phi, X)$ is k-Ore. By Claim 34, $Y = K_k$.

Recall that $\{Q, R\}$ is a partition of $V(J)$. Then by (30) and (31C), $\{Q, X\}$ is a partition of $V(Y)$. Let $t = |X|$ (and $t = |X'|$ also by (31C)). By (31A), $t \geq 2$. Because $|R| \leq n-2$, $|Q| \geq 2$. So since $V(Y) = X \cup Q$, we have $t \leq k-2$. Then J is obtained from $J[R]$ by adding $k-t$ vertices and at least $\binom{k}{2} - \binom{t}{2}$ edges (a vertex in Q may be adjacent to more than one vertex in a color class of ϕ). So

$$\rho_k(V(J)) \leq \rho_k(R) + (k-t)(k+1)(k-2) - \left(\binom{k}{2} - \binom{t}{2}\right) 2(k-1). \tag{32}$$

We know that $2 \leq t \leq k-2$. We will show that $t = 2$. This is clear when $k = 4$. Let $k \geq 5$. Denote the RHS of (32) by $\mu(k, t, R)$. For fixed k and R, $\mu(k, t, R)$ is quadratic in t with a positive coefficient at t^2. So, if $3 \leq t \leq k-2$, then $\mu(k, t, R) \leq \max\{\mu(k, 3, R), \mu(k, k-2, R)\}$. Furthermore,

$$\begin{aligned}\mu(k, k-2, R) &= \rho_k(R) + 2(k+1)(k-2) - k(k-1)^2 \\ &\quad + (k-1)(k-2)(k-3) \\ &\leq 2(k-1)(k-2) - 2k^2 + 8k - 10 = 2k - 6 \leq y_k,\end{aligned}$$

and when $k \geq 5$,

$$\begin{aligned}\mu(k, 3, R) &= \rho_k(R) + (k-3)(k+1)(k-2) - k(k-1)^2 + 6(k-1) \\ &\leq 2(k-1)(k-2) - 2k^2 + 6k = 4 \leq y_k.\end{aligned}$$

Since $\rho_k(V(J)) > y_k$ by the choice of J, we conclude that $t = 2$ and $J[Q] = K_{k-2}$. Moreover, $\mu(4, 2, R) \leq 2(4-1)(4-2) + 20 - (6-1)6 = 2 = y_4$, so $k \geq 5$.

Thus by (32),

$$\begin{aligned}\rho_k(V(J)) &\le \mu(k,2,R)\\ &= \rho_k(R) + (k-2)^2(k+1) - k(k-1)^2 + 2(k-1)\\ &= \rho_k(R) - k^2 + k + 2.\end{aligned} \tag{33}$$

Hence by (33) and (12),

$$\rho_k(R) \ge \rho_k(V(J)) + k^2 - k - 2 \ge y_k + k^2 - k.$$

Plugging in the values of y_k, we get

$$\rho_k(R) \ge 2(k-1)(k-2)-2 \text{ for } k \ge 6 \text{ and } \rho_5(R) \ge 2(5-1)(5-2) = 24. \tag{34}$$

Since for $k \ge 5$, $2(k-1)(k-2)-2 > y_k + 4(k-1)$, we have

$$i \ge 2. \tag{35}$$

Also we conclude that each $v \in Q$ has exactly two neighbors in R, since otherwise the upper bound on $\rho_k(V(J))$ in (32) and (33) would be stronger by $2(k-1)$, and this combined with (23) would lead to

$$\rho_k(V(J)) \le -2(k-1) + 2(k-1)(k-2) - k^2 + k + 2 = k^2 - 7k + 8 \le y_k.$$

Let $Q = \{v_1, \ldots, v_{k-2}\}$ and let $N(v_j) \cap R = \{u_{j,1}, u_{j,2}\}$ for $j = 1, \ldots, k-2$. If there exists a proper $(k$–$1)$-coloring ϕ' of $J[R]$ such that $\phi'(u_{j,1}) = \phi'(u_{j,2})$ for some j, then ϕ' may be extended to all of J greedily by first coloring $Q - v_j$ and at the end coloring v_j (at each step at most k–2 colors must be avoided). Similarly, if $\{\phi'(u_{j,1}), \phi'(u_{j,2})\} \ne \{\phi'(u_{j',1}), \phi'(u_{j',2})\}$ for some $j \ne j'$, then ϕ' may be extended to all of J greedily by first coloring $X - v_j - v_{j'}$ and at the end coloring v_j and $v_{j'}$. Thus for any proper $(k-1)$-coloring ϕ' of $J[R]$,

$$\begin{gathered}\text{for all } 1 \le j, j' \le k-2,\\ \phi'(u_{j,1}) \ne \phi'(u_{j,2}) \text{ and } \{\phi'(u_{j,1}), \phi'(u_{j,2})\} = \{\phi'(u_{j',1}), \phi'(u_{j',2})\}.\end{gathered} \tag{36}$$

Because $3 \le s = |R_*|$ by Claim 21, there exist distinct vertices $v', v'' \in Q$ such that $N(v') \cap R \ne N(v'') \cap R$. By symmetry, we may assume $u_{1,1} \notin N(v_2)$. Let J^* be obtained from $J[R]$ by adding edges $e_1 = u_{1,1}u_{2,1}$ and $e_2 = u_{1,1}u_{2,2}$ when they do not exist. By (36), $\chi(J^*) \ge k$, which implies that e_1 or e_2 was not in J. Thus J^* contains a k-critical subgraph J°. By the minimality of J and the fact that $|J^\circ| \le |R| \le |J|-2$, graph J° is k-Ore or $\rho_{k,J^*}(V(J^\circ)) \le y_k$. Because $J^\circ = J[V(J^\circ)] + S$ for some nonempty $S \subseteq \{e_1, e_2\}$, we know that $J[V(J^\circ)]$ is not a clique. If $V(J^\circ)$ satisfies (23), then the minimality of i applied to (24) states that $\rho_{k,J}(V(J^\circ)) > y_k + 2i(k-1)$; if $V(J^\circ)$ does not satisfy (23), then (as $\chi(J^\circ)=k$ implies $|V(J^\circ)|\geq k$, and the other conditions are proved above)

$$\rho_{k,J}(V(J^\circ)) > 2(k-1)(k-2) \geq y_k + 2i(k-1).$$

Since $i\geq 2$ by (35) and because we have added at most two edges, by (24), $\rho_{k,J^*}(V(J^\circ)) \geq \rho_{k,J}(V(J^\circ)) - 4(k-1) > y_k$, and so J° is k-Ore. Recall by Fact 15.9, the vertex sets of k-Ore graphs have potential $k(k-3)$. So, $\rho_{k,J^*}(V(J^\circ))\leq k(k-3)$ and our above inequality becomes by Fact 11.8

$$\rho_{k,J}(V(J^\circ)) \leq \rho_{k,J^*}(V(J^\circ))+4(k-1) \leq k(k-3)+4(k-1) \leq y_k+3(2(k-1)).$$

Hence $V(J^\circ)$ satisfies (24) for some $i\leq 2$. By (35), we have $i=2$. By the minimality of i and of $|R|$, this gives

$$i=2, V(J^\circ)=R \text{ and } J[R]=J^\circ - e_1 - e_2. \tag{37}$$

Also, since $i=2$ and $(k+1)(k-2)\leq y_k+2(2(k-1))$, the minimality of the potential of R in (24), and Fact 11.(2,3,7) imply

(38) $J[R]$ contains no set with potential at most $(k+1)(k-2)$.

For all $S\subseteq V(J^\circ)-u_{1,1}$, we have $J^*[S]\cong J[S]$. Thus, by (37) and (38), Claim 20 applies to $J^\circ=J^*$ and $u_{1,1}$. By this claim, $J^\circ-u_{1,1}$ contains a clique S of order $k-1$ such that each vertex in S has degree $k-1$. Since $u_{1,1}\notin S$, S is also a clique in J. Since $e_1,e_2\subset N(Q)=R_*$, if $u\in S-N(Q)$, then $d_J(u)=k-1$. Because $N(Q)$ is 2-colorable (by (36)), this implies that there is an $S'\subset S$ with $|S'|\geq k-3$ such that $d_J(u)=k-1$ for all $u\in S'$. Each vertex of S' is in a cluster by Fact 32, and Claim 33 says that all of S' is one cluster and that $|S'|=k-3$. Let $\{u'\}=N(S')-S$. Then by Fact 11 (parts 1, 2, and 4)

$$\begin{aligned}\rho_{k,J}(S\cup u') &\leq \rho_{k,J}(S)+\rho_{k,J}(\{u'\})-2(k-1)(k-3)\\ &= k^2+k-4 < y_k+3(2(k-1)).\end{aligned}$$

By the minimality of R, we have $R=S\cup\{u'\}$. So $u_{1,1}=u'$ and $J[R]$ is a k-clique minus the edges $e_1=u_{1,1}u_{2,1}$ and $e_2=u_{1,1}u_{2,2}$. But then for any possible choice of $u_{1,2}$, there exists a $(k-1)$-coloring ϕ of $J[R]$ such that $\{\phi'(u_{1,1}),\phi'(u_{1,2})\}\neq\{\phi'(u_{2,1}),\phi'(u_{2,2})\}$. This contradiction to (36) finishes Case 1.

In all subsequent cases, we will use Lemma 37 in order to construct either a $(k-1)$-coloring of J or a $(k-1)$-coloring of $J[R]$ fitting into Case 1. Lemma 37 uses variables i',z,s,w. We will use several different values for i', but z,s,w are always defined by the discussion leading to (26), and by Claim 21, $s\geq 3$.

Case 2: $2i \geq z = |E(R,Q)|$. By (25) and (26), in order to have $2i \geq |E(R,Q)|$, we need $i = \frac{k}{2}$, $k \geq 6$, and $|E(R,Q)| = k$. For $k \geq 6$, we know that $y_k = k^2 - 5k + 2$. By Lemma 37 with $i' = i-1$, we can add to $J[R_*]$ a set E_1 of at most $i-1$ edges such that (21) holds with $i' = i-1$. Let $H_1 = J[R] \cup E_1$. By (24),

$$\rho_{k,H_1}(R') > y_k + 2i(k-1) - (i-1)2(k-1) = k(k-3)$$

for every $R' \subseteq R$ with $|R'| \geq 2$ (as this inequality is also true for K_{k-1}). So, by Corollary 9, H_1 has a proper $(k-1)$-coloring ϕ. Moreover, ϕ is also a proper $(k-1)$-coloring of $J[R]$. Since Case 1 does not hold, ϕ has a color class C that satisfies neither (27) nor (28). This means that $\sum_{u \in R_* - C} w(u) = i-1$ (the lower bound is from (21)) and $\sum_{u \in C} w(u) \geq k-1$. But then

$$|E(R,Q)| \geq k - 1 + i - 1 \geq k - 1 + \frac{k}{2} - 1 = \frac{3k}{2} - 2.$$

Since $k \geq 6$, this contradicts $|E(R,Q)| = k$. This concludes Case 2.

If Case 2 does not hold, then $z > 2i$ and, since $s = |R_*| \geq 3$ by Claim 21, the "moreover" part of Lemma 37 holds when applied with $i' = i$ (recall the calculation right after (24) showing that $i \geq 1$). The cases below analyze the possibilities in Lemma 37.

Case 3: The set $\{w(u_1), \dots, w(u_s)\}$ is i-special: $s = i+1$ and $w(u_2) = \dots = w(u_s) = 1$. This means that many (exactly $z - i \geq i$) edges connect u_1 with Q and each of the vertices $u_2, \dots, u_{i+1}$ is connected to Q by exactly one edge. For $j = 2, \dots, i+1$, let q_j be the vertex in Q such that $u_j q_j \in E(J)$. Let $E_0 = \{u_1 u_j \colon 2 \leq j \leq i\}$ and $H_0 = J[R] \cup E_0$. Since $|R| < n$, H_0 is smaller than J. Since $|E_0| = i-1$, by (24),

$$\rho_{k,H_0}(R') > y_k + 2i(k-1) - (i-1)2(k-1) \geq k(k-3)$$

for every $R' \subseteq R$ with $|R'| \geq 2$ (as such an inequality is also true for K_{k-1}). So, by the second part of Corollary 9, H_0 has a proper $(k-1)$-coloring ϕ. By construction, ϕ is a proper $(k-1)$-coloring of $J[R]$ that satisfies $\phi(u_j) \neq \phi(u_1)$ for each $2 \leq j \leq i$. If $\phi(u_{i+1}) \neq \phi(u_1)$, then for every monochromatic subset M of R_* in $J \cup E_0$ with $|M| \geq 2$, (21) holds. But this coloring satisfies (27), which contradicts that Case 1 does not apply, so suppose $\phi(u_{i+1}) = \phi(u_1)$.

Let J_0 be obtained from $J[V(J) - (R - u_1)]$ by adding edge $u_1 q_{i+1}$. By Claim 28, J_0 has a $(k-1)$-coloring ϕ'. Since $i \leq \frac{k}{2}$ by (25), we can rename the colors in ϕ' so that $\phi'(u_1) = \phi(u_1) = \phi(u_{i+1})$ and $\phi(\{u_2, \dots, u_i\}) \cap \phi'(\{q_2, \dots, q_i\}) = \emptyset$. Then $\phi \cup \phi'$ is a proper $(k-1)$-coloring of J, a contradiction.

Case 4: The set of weights $\{w(u_1),\dots,w(u_s)\}$ is not i-special and $2i<z$, so that Part (i) or (ii) of the "moreover" part of Lemma 37 holds when applied with $i'=i$. If Part (i) holds, then we take this set E_0 of at most $i-1$ edges and let $H_0=J[R]\cup E_0$. In this case by (24), $\rho_{k,H_0}(R')>y_k+2k-2\geq k(k-3)$ for every $R'\subseteq R$ with $|R'|\geq 2$ (as such an inequality also holds for K_{k-1}). So, by the second part of Corollary 9, H_0 has a $(k-1)$-coloring ϕ, satisfying (27) of Case 1. Suppose now that Part (ii) holds: *there is a hypergraph H with at most $i-1$ graph edges and a 3-edge $e_0=\{u,v,w\}$ such that $d_H(u_j)\leq w(u_j)$ for all $j=1,\dots,s$ and (21) holds.* Let H_1 be obtained from $J[R]$ by adding the set of edges $E(H)-e_0$ and edge uv. Since $|R|<n$, H_1 is smaller than J. A proper $(k-1)$-coloring of H_1 would satisfy (27) of Case 1, so $\chi(H_1)\geq k$. Then H_1 has a k-critical subgraph H_1'.

Let $R'=V(H_1')$. Because Part (i) of the "moreover" does not hold, $\{u,v\}\subseteq R'$ and $uv\notin J[R']$. So $J[R']$ is not a clique. If H_1' is not a k-Ore graph, then by the minimality of J, $\rho_{k,H_1}(R')\leq y_k$ and so $\rho_{k,J}(R')\leq y_k+2i(k-1)$, contradicting in (24) the minimality of i (as $y_k+2i(k-1)\leq 2(k-1)(k-2)$ and $J[R']$ is not a clique). Thus, H_1' is a k-Ore graph and $\rho_{k,H_1}(R')=k(k-3)\leq y_k+2k-2$. Then

$$\rho_{k,J}(R')\leq\rho_{k,H_1}(R')+2i(k-1)\leq y_k+2(i+1)(k-1),$$

and by the minimality of R, $R'=R$. Furthermore, if $H_1'\neq H_1$, then it has the same vertex set as H_1 and at least one fewer edge, in which case,

$$\begin{aligned}\rho_{k,J}(R')&\leq\rho_{k,H_1'}(R')+2i(k-1)\leq\rho_{k,H_1}(R')+2(i-1)(k-1)\\&\leq k(k-3)+2(i-1)(k-1)\leq y_k+2i(k-1),\end{aligned}$$

a contradiction to (24). So, $H_1'=H_1$, H_1 is a k-Ore graph and so $\rho_{k,J}(R)=k(k-3)+2i(k-1)$.

Recall that $\rho_{k,J}(R)\leq 2(k-1)(k-2)$ by the last inequality of (23), so

$$i\leq\frac{2(k-1)(k-2)-k(k-3)}{2(k-1)}=\frac{1}{2}(k-2+\frac{2}{k-1}).$$

Because i is an integer, this inequality is strict, and so $\rho_{k,J}(R)<2(k-1)(k-2)$. Therefore $\rho_{k,J}(W)>\rho_{k,J}(R)$ if $J[W]=K_{k-1}$. By the minimality of i and R, any $W\subset R$ such that $|W|\geq 2$ and $J[W]\neq K_{k-1}$ satisfies $\rho_{k,J}(W)>\rho_{k,J}(R)$. Graph H_1-uv is $J[R]$ plus $i-1$ edges, so for any $W\subset V(H_1')$ with $|W|\geq 2$ we have

$$\rho_{k,H_1-uv}(W)=\rho_{k,J}(W)-2(i-1)(k-1)>k(k-3)+2(k-1)=(k+1)(k-2).$$

Thus by Lemma 19, H_1-uv has a $(k-1)$-coloring ϕ with $\phi(w)\neq\phi(u)$. This is a $(k-1)$-coloring of $J[R]$ that satisfies (27) of Case 1. ∎

Recall that a standard set has potential $(k+1)(k-2)$. Because $(k+1)(k-2) < 2(k-1)(k-2)$ when $k \geq 4$, Lemma 38 implies:

Corollary 39. *For each $R \subsetneq V(J)$ with $2 \leq |R| \leq n-2$, $\rho_k(R) > (k+1)(k-2)$. In particular, J does not contain a standard set of size at most $n-2$.*

Claim 40. *If v is not in a $(k-1)$-clique X, then $|N(v) \cap X| \leq \frac{k-1}{2}$. Furthermore, if T is a cluster in a $(k-1)$-clique X, then $|T| \leq \frac{k-1}{2}$.*

Proof. If $|N(v) \cap X| \geq \lceil k/2 \rceil$, then $\rho_k(X+v) \leq 2(k-2)(k-1) - 2$. Since $n \geq k+2$, this contradicts Lemma 38. This proves the first part.

Suppose now that T is a cluster in a $(k-1)$-clique X. Since $|X| = k-1$ and $d(w) = k-1$ for every $w \in T$, each such w has the unique neighbor $v(w)$ outside of X. But by the definition of a cluster, $v(w)$ is the same, say v, for all $w \in T$. This means that $T \subseteq X \cap N(v)$, so $|N(v) \cap X| \geq |T|$. Thus the second part follows from the first. ∎

Claim 41. *Suppose T is a cluster in J, $t = |T| \geq 2$, and $N(T) \cup T$ contains a $(k-1)$-clique X. Then $d_J(v) \geq k-1+t$ for every $v \in X - T$.*

Proof. By the definition of a cluster, $|T \cup N(T)| = k$. So, since $|T| \geq 2$ and $|X| = k-1$, $T \cap X \neq \emptyset$. So by Claim 33(ii), $T \subseteq X$, and by Claim 33(i), X contains only one nonempty cluster, namely, T.

Suppose $v \in X - T$ and $d(v) \leq k-2+t$. By the previous paragraph, v is not in a cluster and thus by Fact 32, $d(v) \geq k$. By Claim 30, T is contained in at most one $(k-1)$-clique (which is X), and so

$$N(T) \cup T - v \text{ does not contain } K_{k-1}. \tag{39}$$

Because T and v are parts of the same clique, $|N(v) - T| = d(v) - |T|$, and this is at most $k-2$, since $d(v) \leq k-2+t$. Let $u \in T$ and $J' = J - v + u'$, where u' is a new vertex that satisfies $N[u'] = N[u]$. Suppose J' has a $(k-1)$-coloring $\phi' \colon V(J') \to C = \{c_1, \ldots, c_{k-1}\}$. Then we find a $(k-1)$-coloring ϕ of J as follows: set $\phi|_{V(J)-T-v} = \phi'|_{V(J')-T-u'}$, $\phi(v) \in C - \phi'(N(v) - T)$, and then color T using colors in $\phi'(T \cup u') - \phi(v)$. This is a contradiction, so there is no $(k-1)$-coloring of J'. Thus J' contains a k-critical subgraph J''. Let $W = V(J'')$. By Corollary 9, $\rho_{k,J'}(W) \leq k(k-3)$.

By the criticality of J, graph J'' is not a subgraph of J. So $u' \in W$ and u' has at least $k-1$ neighbors in W. By symmetry, we have $T \subset W$. But then

$$\rho_{k,J}(W - u') \leq k(k-3) - (k-2)(k+1) + 2(k-1)(k-1) = 2(k-2)(k-1).$$

By Lemma 38, this implies that either $J[W-u']$ is a K_{k-1} or $W - u' = V(J) - v$. If the former holds, then because $J[W-u']$ is a complete graph and $T \subset W-u'$ we have $N(T)\cup T \supset J[W-u'] \cong K_{k-1}$, and because $v \notin W$ this is a contradiction to (39). If the latter holds, then we have a contradiction to Fact 22, since $d(v) \geq k$. ∎

Claim 42. *Suppose $xy \in E(J)$, $N[x] \neq N[y]$, x is in a cluster of size s, y is in a cluster of size t, and $s \geq t$. Then x is in a $(k-1)$-clique. Furthermore, $t=1$.*

Proof. Assume that x is not in a $(k{-}1)$-clique. Let $J' = J-y+x'$, where x' is a new vertex with $N_{J'}(x') = N_J[x]-y$. By the definition of a cluster, $d(x) = d(y) = k-1$. Both J' and J have the same number of vertices and the same number of edges (because $xy \in E(J)$, vertex x lost an edge to y and gained an edge to x'). If distinct $v, w \in V(J)-y$, then they have the same closed neighborhood in J' if and only if they have the same closed neighborhood in J. By the definition of a cluster, there are exactly $t-1$ vertices in $V(J)-y$ that have the same closed neighborhood as y. By construction, there are exactly s vertices in $V(J')-x'$ that have the same closed neighborhood in J' as x'. So since $s \geq t$, by Rule (S3), J' is smaller than J.

If J' has a $(k-1)$-coloring $\phi' \colon V(J') \to C = \{c_1, c_2, \ldots, c_{k-1}\}$, then we extend it to a proper $(k-1)$-coloring ϕ of J as follows: define $\phi|_{V(J)-x-y} = \phi'|_{V(J')-x-x'}$, then choose $\phi(y) \in C-(\phi'(N(y)-x))$, and $\phi(x) \in \{\phi'(x), \phi'(x')\} - \{\phi(y)\}$.

So, $\chi(J') \geq k$ and J' contains a k-critical subgraph J''. Let $W = V(J'')$. By the criticality of J and because $y \notin J''$, we have $J'' \neq J$, and J'' is not a subgraph of J. This yields $x' \in W$. By symmetry, $x \in W$. By Corollary 9, $\rho_{k,J'}(W) \leq k(k-3)$. Because of this and the fact that $d(x') = k-1$,

$$(40)\quad \rho_{k,J}(W-x') \leq k(k-3) - \rho_{k,J'}(\{x'\}) + 2(k-1)d(x') = 2(k-2)(k-1).$$

By assumption, x is not in a $(k$–$1)$-clique, so Lemma 38 implies that $|W - x'| > n-2$. Since $y \notin W$, this yields $|W-x'| = n-1$, which implies $V(J') = V(J'')$ and $W-x' = V(J)-y$. By Corollary 9, $\rho_{k,J''}(W) \leq k(k-3)$. Moreover, because J'' is smaller than or equal to J' which is smaller than J, the minimality of J implies that if J'' is not k-Ore, then $\rho_{k,J''}(W) \leq y_k$. In this case we can replace the term $k(k-3)$ in (40) with y_k (as J'' is a subgraph of J' and so $\rho_{k,J'}(W) \leq \rho_{k,J''}(W)$) and get $\rho_{k,J}(W-x') \leq y_k + k^2 - 3k + 4$ contradicting Fact 22. If $J'' \neq J'$, then $\rho_{k,J''}(W) - 2(k-1) \geq \rho_{k,J'}(W)$, and instead of (40) we obtain $\rho_{k,J}(W-x') \leq 2(k-2)(k-1) - 2(k-1)$, again contradicting Fact 22. So $J'' = J'$ and J'' is k-Ore, thus J' is a k-Ore graph.

Since $n > k$, $J' \neq K_k$. Let the separating set $\{u, v\}$, vertex subsets $A = A(J', u, v)$ and $B = B(J', u, v)$, and graphs $\widetilde{J}'(u,v)$ and $\check{J}'(u,v)$ be as in

Fact 14. By Corollary 39, the standard set A is not contained in $V(J)$. Hence $x' \in A$. Therefore $x' \notin V(\check{J}'(u,v)) - u*v$. Thus for every $W \subseteq V(\check{J}') - u*v$ with $|W| \geq 2$, by Corollary 39, we have $\rho_{k,\check{J}'(u,v)}(W) = \rho_{k,J}(W) > (k+1)(k-2)$. Then by Claim 20, there exists an

$$S \subseteq V(\check{J}'(u,v)) - u*v \text{ such that} \\ \check{J}'(u,v)[S] \cong K_{k-1}, \text{ and } d_{\check{J}'(u,v)}(w) = k-1 \text{ for all } w \in S. \tag{41}$$

By Claim 30, vertex y in J is adjacent to at most $k-3$ vertices in S. Because $V(J) - y \subset V(J')$, (41) yields that vertices in $S - N(y)$ have degree $k-1$ in J. So by Fact 32, S contains a $V(J)$-cluster T, which by Claim 33 contains all vertices in S of degree $k-1$ and $|T| \geq |S - N(y)| \geq 2$. Then by Claim 41, the degree of each vertex in $S - T$ in J is at least $k+1$. This is impossible, since by Fact 15.5 each of them has in J at most one extra neighbor (and it is y) in comparison with $\check{J}'(u,v)$, where their degree was only $k-1$. This proves the first part: x is in a $(k-1)$-clique, say X.

Let T_y be the cluster containing y. By the definition of a cluster, every vertex in T_y has the same neighbors as y, and so $T_y \subseteq N(x)$. The $(k-1)$-clique X containing x is a part of $N[x]$. By Claim 33(ii) if $T_y \cap X \neq \emptyset$, then $T_y \subseteq X$, and by Claim 33(i) $y \in T_y - X$; so $T_y \cap X = \emptyset$. Therefore $|T_y| \leq |N(x) - X| = d(x) - (k-2) = 1$. This proves the second part. ∎

Claim 43. *Suppose T is a cluster in J, $t = |T| \geq 2$, and $N(T) \cup T$ does not contain K_{k-1}. Then $d_J(v) \geq k-1+t$ for every $v \in N(T) - T$.*

Proof. By Claim 42, $d(v) \geq k$. Now the proof follows exactly as the proof to Claim 41. ∎

5. Proof of Theorem 10

Now we are ready to prove the theorem. Recall that in Section 3 we defined a relation "smaller," and J is a "smallest" counterexample to Theorem 10: it is a k-critical graph with $\rho_k(V(J)) > y_k$ and is not k-Ore.

We will use the following result on k-critical graphs which is Corollary 9 in [26].

Lemma 44 ([26]). *Let J be a k-critical graph. Let disjoint vertex subsets A and B be such that*

(a) at least one of A an B is independent;
(b) $d(a) = k-1$ for every $a \in A$;
(c) $d(b) = k$ for every $b \in B$;
(d) $|A| + |B| \geq 3$.

Then (i) $e(J(A,B)) \leq 2(|A|+|B|) - 4$ and (ii) $e(J(A,B)) \leq |A| + 3|B| - 3$.

5.1. Case $k=4$

In this subsection we prove the theorem for $k=4$. Specifically, we will prove that $|E(J)|\geq \frac{5}{3}|V(J)|$, which will imply that $\rho_{4,J}(V(J))\leq 0\leq\ y_4=2$.

Claim 45. *Each vertex with degree* 3 *has at most one neighbor with degree* 3.

Proof. Let x be such that $N(x)=\{a,b,c\}$ and $d(a)=3$. Then x and a are each in a cluster. Because no cluster is larger than $k-3=1$ by Claim 33, a and x are in different clusters. Then by Claim 42, $J[\{x,b,c\}]$ is a K_3. So by Claim 33, $d(b),d(c)\geq 4$. ∎

We now use discharging to show that $|E(J)|\geq \frac{5}{3}n$. Each vertex begins with charge equal to its degree. If $d(v)\geq 4$, then v gives charge $\frac{1}{6}$ to each neighbor. Note that v will be left with charge at least $\frac{5}{6}d(v)\geq\frac{10}{3}$. By Claim 45, each vertex of degree 3 will end with charge at least $3+\frac{2}{6}=\frac{10}{3}$. Therefore, the total charge is at least $\frac{10}{3}n$, and thus so is the sum of the vertex degrees. Hence the number of edges is at least $\frac{5}{3}n$. ∎

5.2. Case $k=5$

In this subsection we prove the theorem for $k=5$. Specifically, we will prove that $|E(J)|\geq \frac{9}{4}|V(J)|$, which will imply that $\rho_{5,J}(V(J))\leq 0<y_5=4$.

Claim 46. *If $k=5$, then each cluster has only one vertex.*

Proof. Suppose the claim does not hold. By Claim 33, every cluster has size at most $k-3=2$, so assume that $\{x,y\}$ is a cluster: $N[x]=N[y]$ and $d(x)=d(y)=4$. Let $N(x)=\{y,a,b,c\}$. By assumption J is not 5-Ore and therefore $J\neq K_5$ (and since J is critical, it does not contain a K_5). By Claim 28, J does not contain a subgraph isomorphic to $K_5{-}e$. Therefore, any five vertices in J induce at most $\binom{5}{2}{-}2$ edges, and thus $|E(J[\{a,b,c\}])|\leq 1$. By Claims 41 and 43, we can rename the vertices in $\{a,b,c\}$ so that $ab,ac\notin E(J)$ and $d(c)\geq 6$.

We obtain J' from J by deleting x and y and gluing a with b. If J' is 4-colorable, then so is J. This is because any 4-coloring of J' will have at most 2 colors on $N[x]-\{x,y\}$ and therefore could be extended greedily to x and y.

So J' contains a k-critical subgraph J''. Let $W'=V(J'')$. Then by Corollary 9, $\rho_{5,J'}(W')\leq\rho_{5,J''}(W')\leq 10$. Furthermore, because J'' is smaller than J,

$$\text{if } J'' \text{ is not } k\text{-Ore, then } \rho_{5,J''}(W')\leq 4. \tag{42}$$

Because J is critical and $x,y \notin J'' \subseteq J'$, graph J'' is not a subgraph of J. This implies that $a*b \in V(J'')$. Let $W = W' - a*b + a + b + x + y$. By the definition of potential,

(43)
$$\rho_{5,J}(W) = \rho_{5,J''}(W') + 18\left(|W| - |W'|\right) - 8\left(|E(J[W])| - |E(J''[W'])|\right).$$

If $c \notin W'$, then by construction, W has 3 more vertices and induces at least 5 more edges than W'. If $c \in W'$, then W has 3 more vertices and induces at least 7 more edges than W'. Suppose first that $c \notin W'$, so that (43) becomes $\rho_{5,J}(W) \le 10 + 54 - 40 = 24$. Because $ab \notin E(J)$, $J[W]$ is not a K_4. So by Lemma 38, $|W| \ge n-1$. Therefore $W = V(J) - c$ and $\rho_{5,J}(W) \le 24$, but this contradicts Fact 22 because $d(c) \ge 6 = k+1$.

So now we assume that $c \in W'$, which means that (43) becomes $\rho_{5,J}(W) \le 10+54-56 \le 8$. By Lemma 27, $W = V(J)$, which then implies $V(J'') = V(J')$. Furthermore, if J'' is not k-Ore, then by (42), $\rho_{5,J''}(W') \le 4$ and the right hand side of (43) has an extra -6. If J'' is a proper subgraph of J', then $|E(J[W])| - |E(J''[W'])|$ is at least 8 instead of the previously applied bound of 7. In either case our bound becomes stronger by at least 6, and becomes $\rho_{5,J}(V(J)) = \rho_{5,J}(W) \le 8 - 6 = 2 < y_k$, a contradiction to the choice of J. So J'' is k-Ore and $J'' = J'$.

This also implies that $N(a) \cap N(b) = \{x,y\}$, because otherwise, since $J'' = J'$, we would have gained an extra edge when we undo the merge of a and b into $a*b$, which would increase $|E(J[W])| - |E(J''[W'])|$ by 1 in (43) yielding the same contradiction.

Since $d(c) \ge 6$, J' cannot be K_5. Let the separating set $\{u,v\}$, vertex subsets $A = A(J',u,v)$ and $B = B(J',u,v)$, and graphs $\widetilde{J}'(u,v)$ and $\check{J}'(u,v)$ be as in Fact 14. By Fact 15.3, $\rho_{k,J'}(A) = (k+1)(k-2)$. By Corollary 39, proper subgraphs of J have strictly larger potential than $(k+1)(k-2)$ and so $a*b \in A$. Therefore $J'[B-u-v] \subset J$ and so $V(\check{J}') - V(J) = u*v$. Hence by Corollary 39, $\rho_{k,\check{J}'(u,v)}(U) = \rho_{k,J}(U) > (k+1)(k-2)$ for every $U \subseteq V(\check{J}') - u*v$ with $|U| \ge 2$. Then by Claim 20, there exists an $S \subseteq V(\check{J}'(u,v)) - u*v$ such that $\check{J}'(u,v)[S] \cong K_{k-1}$, and $d_{\check{J}'(u,v)}(w) = k-1 = 4$ for all $w \in S$.

We claim that

(44) *each vertex in $S - c$ has degree $k-1$ in J.*

Indeed, how it is possible that a vertex $w \in S$ has a larger degree in J than in $\check{J}'$? By Fact 15.7, $d_{\check{J}'}(w) = d_{J'}(w)$. Because $N(a) \cap N(b) = \{x,y\}$, the only vertices whose degree in J could be greater than in J' are a,b,c. But we already showed that $a*b \notin V(\check{J}') - u*v$ and $S \subseteq V(\check{J}'(u,v)) - u*v$, so $S \cap \{a,b,c\} \subseteq \{c\}$. This proves (44).

By Fact 32 and (44), every vertex in $S-c$ is in a cluster. Because S is a $(k-1)$-clique, by Claim 33 there is only one cluster in S, so the claim implies that $T=S-c$ is a cluster and $|T|=3$, which contradicts Claim 33 that each cluster in J has size at most $k-3=2$. ∎

Claim 47. *Each K_4-subgraph of J contains at most one vertex with degree 4. Furthermore, if $d(x)=d(y)=4$ and $xy\in E(J)$, then each of x and y is in a K_4.*

Proof. Each vertex of degree 4 is in a cluster by definition, and by Claim 33, each K_4 contains only one cluster. The first statement of our claim then follows from Claim 46 and the second – from Claim 42. ∎

Definition 48. Let $H\subseteq V(J)$ be the set of vertices of degree 5 not in a K_4, and $L\subseteq V(J)$ be the set of vertices of degree 4 not in a K_4. Set $\ell=|L|$, $h=|H|$ and $e_0=|E(L,H)|$.

Claim 49. $e_0\leq 3h+\ell$.

Proof. This is trivial if $h+\ell\leq 2$. By Claim 47, L is independent. So the claim follows by Lemma 44(ii) with $A=L$ and $B=H$. ∎

We will now use discharging to show that $|E(J)|\geq \frac{9}{4}n$, which will finish the proof to the case $k=5$. Let every vertex $v\in V(J)$ have initial charge $d(v)$. The discharging has one rule:

Rule R1: Each vertex in $V(J)-H$ with degree at least 5 gives charge $1/6$ to each neighbor.

We will show that the charge of each vertex in $V(J)-H-L$ is at least 4.5, and then show that the average charge of the vertices in $H\cup L$ is at least 4.5.

Claim 50. *After discharging, each vertex in $V(J)-H-L$ has charge at least* 4.5.

Proof. Let $v\in V(J)-H-L$. If $d(v)=4$ and $v\notin L$, then v is in a K_4 and by Claim 47 v receives charge $1/6$ from at least 3 neighbors and gives no charge. If $d(v)=5$ and $v\notin H$, then v is in a K_4 and by Claim 47 $N(v)$ contains at least 2 vertices with degree at least 5. Therefore, v gives charge $1/6$ to 5 neighbors, but receives charge $1/6$ from at least 2 neighbors. If $d(v)\geq 6$, then v is left with charge at least $5d(v)/6\geq 4.5$. ∎

Claim 51. *After discharging, the sum of the charges on the vertices in $H\cup L$ is at least $4.5|H\cup L|$.*

Proof. By Claim 47, if $v \in L$, then every vertex in $N(v)$ has degree at least 5. By Rule R1, the vertices in L receive from outside of $H \cup L$ the charge at least $\frac{1}{6}(4\ell - |E(H,L)|)$. By Claim 49, $|E(H,L)| \le 3h + \ell$. So, the total charge on $H \cup L$ is at least

$$5h + 4\ell + \frac{1}{6}(4\ell - (3h + \ell)) = 4.5(h + \ell),$$

as claimed. ∎

Combining Claims 50 and 51, the total charge is at least $\frac{9}{2}n$. Thus, the degree sum of $V(J)$ is at least $\frac{9}{2}n$, and so $|E(J)| \ge \frac{9}{4}|V(J)|$. ∎

5.3. Case $k \ge 6$

In this subsection we prove Theorem 10 for $k \ge 6$. We will prove that $|E(J)| \ge \frac{(k+1)(k-2)}{2(k-1)}|V(J)|$, which will imply that $\rho_{k,J}(V(J)) \le 0 \le y_k = k^2 - 5k + 2$. This proof will involve several claims.

Claim 52. *Suppose $k \ge 6$, X is a $(k-1)$-clique, and $v \in X$ has degree $k-1$. Then X contains at least $(k-1)/2$ vertices with degree at least $k+1$.*

Proof. Let $\{u\} = N(v) - X$. Assume that X contains at least $k/2$ vertices with degree at most k. By Claim 40, $|N(u) \cap X| < k/2$, so there exists a $w \in X$ such that $uw \notin E(J)$ and $d(w) \le k$. By Claim 29, $d(w) = k$, so assume $N(w) - X = \{a, b\}$. Let J' be obtained from $J - v$ by adding edges ua and ub if they do not already exist.

Suppose J' has a $(k-1)$-coloring f. If $f(u)$ is not used on $X - w - v$, then we recolor w with $f(u)$. So, v will have at least two neighbors of color $f(u)$, and we can extend the $(k-1)$-coloring to v.

Thus J' is not $(k-1)$-colorable and so contains a k-critical subgraph J''. Let $W = V(J'')$. By Corollary 9, $\rho_{k,J'}(W) \le k(k-3)$ and so

$$\rho_{k,J}(W) \le k(k-3) + 2(k-1)(2) = k^2 + k - 4 < 2(k-2)(k-1).$$

If $W \ne V(J')$, then this contradicts Lemma 38, since in this case $|W| \le |V(J')| - 1 \le n - 2$. So, $W = V(J')$.

If J'' is not a k-Ore graph, then by the minimality of J, $\rho_{k,J''}(W) \le y_k$, and since adding edges only reduces potential, we have $\rho_{k,J'}(W) \le \rho_{k,J''}(W)$, and so

$$\rho_{k,J}(V(J)) \le \rho_{k,J'}(W) + (k-2)(k+1)(1) - 2(k-1)(k-3) < y_k$$

when $k \ge 6$. This contradicts our choice of J, and so J'' is k-Ore.

Suppose $J'' \neq J'$. Since $W = V(J') = V(J'')$, it follows that $\rho_{k,J'}(W) \leq \rho_{k,J''}(W) - 2(k-1)$, which leads to the same contradiction because by Fact 11.8, $\rho_{k,J''}(W) - 2(k-1) \leq k(k-3) - 2(k-1) \leq y_k$. So, our case is that J'' is a k-Ore graph and $J'' = J'$, i.e., J' is a k-Ore graph. Therefore $\rho_{k,J''}(W) = k(k-3)$ by Fact 11.10.

Since $J' - ua - ub$ is a subgraph of J, by Corollary 39, $\rho_{k,J'}(U) = \rho_{k,J}(U) > (k+1)(k-2)$ for every $U \subseteq V(J') - u$ with $|U| \geq 2$. Then by Claim 20, there exists an

(45)
$$S \subseteq V(J') - u \text{ such that } J'[S] \cong K_{k-1}, \text{ and } d_{J'}(v) = k-1 \text{ for all } v \in S.$$

Note that for every $z \in S - a - b - N_J(v)$, we have $d_J(z) = d_{J'}(z) = k-1$.

Suppose $z \in S \cap N_J(v)$. Then since $N_J(v) = \{u\} \cup X$ and $u \notin S$, we have $z \in X \cap S$. So $z \notin \{a, b\}$ and therefore $N_{J'}(z) = N_J(z) \cup \{v\}$, and thus $d_J(z) \geq |X \cup S| - 1$. Because $|S| = k-1$, there exists a $z' \in S - X$. By construction of J' and because S is a clique, we have that $X \cap S \subseteq N_J(z') \cap X$. By Claim 40, $|X \cap S| \leq |N_J(z') \cap X| \leq (k-1)/2$. So for $k \geq 6$,

$$d_{J'}(z) = d_J(z) - 1 \geq |X \cup S| - 2 \geq \left\lceil \frac{k-1}{2} \right\rceil + (k-1) - 2 \geq k,$$

a contradiction to (45). Thus, $S \cap N(v) = \emptyset$.

By Fact 32 and Claim 33, the vertices in $S - a - b$ are part of one cluster in J with size at least $k - 1 - 2$ in the $(k-1)$-clique S, which contradicts Claim 40 since $k - 3 > \frac{k-1}{2}$ for $k \geq 6$. ∎

Claim 53. *If $k = 6$ and a cluster T is contained in a 5-clique X, then $|T| = 1$.*

Proof. By Claim 40, assume that $T = \{v_1, v_2\}$. Let $N(v_1) - X = \{y\}$ and $T - X = \{u, u', u''\}$. By Claim 52, $d(u), d(u'), d(u'') \geq 7$. By Claim 40, $|N(y) \cap X| < k/2$, and by the definition of a cluster, $\{v_1, v_2\} \subset N(y)$. Thus, $N(y) \cap \{u, u', u''\} = \emptyset$. Obtain J' from $J - T$ by gluing u to y.

Suppose that J' has a 5-coloring. Then we can extend this coloring to a 5-coloring of J by greedily assigning colors to T, because only 3 different colors appear on the set $\{u, u', u'', y\}$. So we may assume that $\chi(J') \geq 6$. Then J' contains a 6-critical subgraph J''. Let $W = V(J'')$. Then by Corollary 9, $\rho_{6,J'}(W) \leq 6(6-3) = 18$. Since J'' is not a subgraph of J because J itself is critical, $u * y \in W$. Let $t = |\{u', u''\} \cap W|$.

Case 1: $t = 0$. Then

$$\rho_{6,J}(W - u * y + y + X) \leq 18 + 28(5) - 10(12) = 38, \tag{46}$$

By Lemma 38, $|W-u{*}y+y+X|\geq n-1$. The bound (46) did not account for the edges in $E(\{u',u''\},V(J)-X)$, but each of u',u'' has at least 3 neighbors outside of X. Thus, we strengthen (46) to $\rho_{6,J}(W-u{*}y+y+X)\leq 38-10\cdot 4<0$.

Denote $R=W-u*y+y+u+T$.

Case 2: $t=1$. Then $\rho_{6,J}(R)\leq 18+28(3)-10(7)=32$. By Lemma 38, this yields $|R|\geq n-1$, so R is either $V(J)-u'$ or $V(J)-u''$. But because $d(u'),d(u'')\geq 7=k+1$, Fact 22 says that $\rho_{6,J}(V(J)-u'),\rho_{6,J}(V(J)-u'')>y_k+k^2+k=50$, a contradiction.

Case 3: $t=2$. Then $|R|\geq 4$ and

$$\rho_{6,J}(R)=\rho_{k,J''}(W)+28\left(|R|-|W|\right)-10\left(|E(J[R])|-|E(J''[W])|\right)\\ \leq 18+28(3)-10(9)=12. \tag{47}$$

Since $|R|\geq 4$, by Lemma 38, $|R|\geq n-1$. By Fact 22, every $(n-1)$-element set has potential more than $y_k+k^2-3k+4=26$. So $|R|=n$ and $R=W-u*y+y+u+T=V(J)$.

If J'' is not k-Ore, then by the minimality of J, $\rho_{k,J''}(W)\leq y_k=8$. In this case, we can replace in (47) the upper bound of 18 on $\rho_{k,J''}(W)$ with 8, which yields $\rho_{6,J}(R)\leq 2<y_6$, a contradiction. Thus J'' is k-Ore.

Similarly, if $J''\neq J'$, then because $V(J')=V(J'')$ we can replace in (47) the term $-10(9)$ with $-10(10)$ as there is an extra edge in $|E(J[R])|-|E(J''[W])|$, which again yields $\rho_{6,J}(R)\leq 2<y_6$. So $J'=J''$.

Since $J''-u*y$ is a subgraph of J, by Corollary 39, $\rho_{k,J'}(U)=\rho_{k,J}(U)>(k+1)(k-2)$ for every $U\subseteq V(J')-u*y$ with $|U|\geq 2$. Then by Claim 20, there exists an $S\subseteq V(J')-u*y$ such that $J'[S]\cong K_5$, and $d_{J'}(v)=5$ for all $v\in S$. By construction, $S\subseteq V(J)$, so $J[S]\cong K_5$. By Fact 32 each vertex with degree $k-1$ in J is in a cluster, by Claim 33 the S-cluster is unique, and by Claim 40, S has at most 2 vertices of degree $k-1$ in J. So there are at least three vertices $z_1,z_2,z_3\in S$ such that $d_J(z_i)>d_{J'}(z_i)$ for $1\leq i\leq 3$. But the vertices with larger degrees in J than in J' could be of only two types: (a) those in $N_J[v_1]=N_J[v_2]=\{y,v_1,v_2,u,u',u''\}$ and (b) those in $N_J(y)\cap N_J(u)$. Because $J''-u*y$ does not contain $T=\{v_1,v_2\}$ (which was deleted), and $u*y\notin S$, we have at most two vertices of type (a). Then we have a vertex of type (b), but in this case we have an extra edge whose potential was not accounted in (47). So in this case we again instead of (47) get $\rho_{6,J}(R)\leq 12-10<y_6$, a contradiction. ∎

Definition 54. We partition $V(J)$ into four classes: L_0, L_1, H_0, and H_1. Let H_0 be the set of vertices with degree k, H_1 be the set of vertices with degree at least $k+1$, and $H=H_0\cup H_1$. Let

$$L=\{u\in V(J)\colon d(u)=k-1\},$$
$$L_0=\{u\in L\colon N(u)\subseteq H\},$$

and

$$L_1=L-L_0.$$

Set $\ell=|L_0|$, $h=|H_0|$ and $e_0=|E(L_0,H_0)|$.

Claim 55. $e_0\leq 2(\ell+h)$.

Proof. This is trivial if $h+\ell\leq 2$. By definition, L_0 is independent. The claim follows by applying Lemma 44(i) for $A=L$ and $B=H$ for $h+\ell\geq 3$. ∎

Let every vertex $v\in V(J)$ have initial charge $d(v)$. Our discharging has two rules:

Rule R1: If a vertex in H_1 has neighbors of degree $k-1$, then it keeps for itself charge $k-2/(k-1)$ and distributes the rest equally among its neighbors of degree $k-1$. Otherwise, it keeps all its charge for itself.

Rule R2: If a K_{k-1}-subgraph X contains exactly s $(k-1)$-vertices adjacent to a $(k-1)$-vertex x outside of X and not in a K_{k-1}, then each of these s vertices gives charge $\frac{k-3}{s(k-1)}$ to x.

Claim 56. *Each vertex in H_1 gives to each neighbor of degree $k-1$ charge at least $\frac{1}{k-1}$.*

Proof. If $v\in H_1$, then v gives to each neighbor charge at least $\psi(d(v)):=\frac{d(v)-k+2/(k-1)}{d(v)}$. Since $\psi(x)$ is monotonically increasing for $x\geq k$, $\psi(d(v))$ is minimized when $d(v)=k+1$. Then each neighbor of v of degree $k-1$ gets charge at least $(1+2/(k-1))/(k+1)=1/(k-1)$. ∎

Claim 57. *Each vertex in L_1 has charge at least $k-2/(k-1)$.*

Proof. Let $v\in L_1$. By Fact 32 every vertex in $L\supseteq L_1$ is in exactly one cluster. Let v be in a cluster C of size t.

By definition, if v gives anything out to some vertex x by Rule R2, then v is in a $(k-1)$-clique X. In this case, since v has $k-2$ neighbors in X, x is the unique neighbor of v outside of X. Also, by Claim 33, $C\subset X$ and by the definition of a cluster, each $w\in C$ is adjacent to x. So

(48) *if v gives anything out by Rule 2,*
then it gives it at most once and for $s=t$.

Case 1: v is in a $(k-1)$-clique X and $t \geq 2$. Since $v \notin H_1$, it can give charge only by Rule R2. Since v is in a $(k-1)$-clique, it may receive charge only by Rule R1.

Recall that $C \subset X$. By Claim 41, each vertex in $X - C$ has degree at least $k-1+t$. Since $t \geq 2$, this means $X - C \subseteq H_1$. Furthermore, each vertex in $X - C$ has at least $k-2-t$ neighbors with degree at least k (the other vertices of X–C). Therefore, each vertex $u \in X$–C gives by Rule R1 charge at least $\frac{d(u)-k+2/(k-1)}{d(u)-k+2+t}$ to each neighbor of degree $k-1$. Note that this function increases as $d(u)$ increases, so the charge is minimized when $d(u) = k-1+t$. It follows that u gives to v charge at least $\frac{t-1+2/(k-1)}{2t+1}$.

So, v receives from the vertices in $X - C$ by Rule R1 at least $(k-1-t)(\frac{t-1+2/(k-1)}{2t+1})$ and in view of (48) gives out by Rule R2 to at most one vertex and with $s = t$. Thus v has charge at least $k - 1 + (k-1-t)\left(\frac{t-1+2/(k-1)}{2t+1}\right) - \frac{k-3}{t(k-1)}$, which we claim is at least $k - 2/(k-1)$. Let

$$g_1(t) = (k-1-t)((t-1)(k-1)+2) - (2t+1)(k-3)\left(1 + \frac{1}{t}\right).$$

We claim that $g_1(t) \geq 0$, which is equivalent to v having charge at least $k - 2/(k-1)$. Let

$$\widetilde{g}_1(t) = (k-1-t)((t-1)(k-1)+2) - (2t+1)(k-3)(3/2).$$

Note that $\widetilde{g}_1(t) \leq g_1(t)$ when $t \geq 2$, so it is enough to show that $\widetilde{g}_1(t) \geq 0$ on the appropriate domain. Function $\widetilde{g}_1(t)$ is quadratic with a negative coefficient at t^2, so it suffices to check its values at the boundaries. They are

$$\widetilde{g}_1(2) = (k-3)(k-6.5)$$

and

$$\begin{aligned} 4\widetilde{g}_1\left(\frac{k-1}{2}\right) &= (k-1)\left((k-3)(k-1)+4\right) - 6k(k-3) \\ &= k^3 - 11k^2 + 29k - 7 \\ &= (k-7)(k^2 - 4k + 1). \end{aligned}$$

Each of these values is non-negative when $k \geq 7$, and if $k = 6$, then the case does not apply by Claim 53.

Case 2: $t \geq 2$ and v is not in a $(k-1)$-clique. Then v cannot give charge by Rule R2 and may receive charge by Rule R1. By Claim 43, each neighbor of v outside of C has degree at least $k-1+t \geq k+1$ and is in H_1. Therefore v has charge at least $k-1+(k-t)\left(\frac{t-1+2/(k-1)}{k-1+t}\right)$. We define

$$\begin{aligned} g_2(t) &= (k-t)\left(t-1+\frac{2}{k-1}\right) - \frac{k-3}{k-1}(k-1+t) \\ &= t(k-t) - 2\left(1-\frac{2}{k-1}\right)(k-1) \\ &= t(k-t) - 2(k-3). \end{aligned}$$

Note that $g_2(t) \geq 0$ is equivalent to v having charge at least $k-2/(k-1)$. The function $g_2(t)$ is quadratic with a negative coefficient at t^2, so it suffices to check its values at the boundaries. They are

$$g_2(2) = 2(k-2) - 2(k-3) = 2$$

and

$$g_2(k-3) = (k-3)(3) - 2(k-3) = k-3.$$

Each of these values is positive.

Case 3: $t=1$. By the definition of L_1, v is adjacent to at least one vertex w with degree $k-1$. Since $|C|=t=1$ and so $C=\{v\}$, $w \notin C$. So by Fact 32, v and w are in different clusters. This means $N[w] \neq N[v]$. If v is not in a $(k-1)$-clique X, then by Claim 42, w is in a $(k-1)$-clique and a cluster of size at least 2. In this case v will receive charge $(k-3)/(k-1)$ in total from the cluster containing w using Rule R2, and will not give away any charge. Therefore we may assume that v is in a $(k-1)$-clique X.

By Claim 52, there exists a $Y \subset X$ such that $|Y| \geq \frac{k-1}{2}$ and every vertex in Y has degree at least k+1. By Claim 33, none of the vertices in X–C = X–$\{v\}$ is in a cluster, which by Fact 32 means that every vertex in $X-\{v\}$ has degree at least k. So each vertex in Y has at least k–3 neighbors with degree at least k (the vertices of X besides v and itself). Therefore, by Rule R1 each vertex $u \in Y$ donates charge at least $g_3(d(u)) = \frac{d(u)-k+2/(k-1)}{d(u)-k+3}$ to each neighbor of degree k–1. Since $d_3(d(u))$ increases as $d(u)$ increases, the charge is minimized when $d(u)=k+1$. It follows that u gives to v charge at least $d_3(k+1) = \frac{1+2/(k-1)}{4}$, and v has charge at least

$$k-1+\frac{k-1}{2}\left(\frac{1+2/(k-1)}{4}\right) = k+\frac{k-7}{8},$$

which is at least $k-2/(k-1)$ when $k \geq 6$. ∎

By the discharging rules and Claims 56 and 57, after discharging,

a) the charge of each vertex in $H_1 \cup L_1$ is at least $k-2/(k-1)$;
b) the charges of vertices in H_0 did not decrease;
c) along every edge from H_1 to L_0 the charge at least $1/(k-1)$ is sent.

Thus by Claim 55, the total charge F of the vertices in $H_0 \cup L_0$ is at least

$$kh + (k-1)\ell + \frac{1}{k-1}\left(\ell(k-1) - e_0\right)$$
$$\geq k(h+\ell) - \frac{1}{k-1}2(h+\ell) = (h+\ell)\left(k - \frac{2}{k-1}\right),$$

and so by a), the total charge of all the vertices of J is at least $n\left(k-\frac{2}{k-1}\right)$. Therefore, the degree sum of J is at least $n\left(k-\frac{2}{k-1}\right) = \left(\frac{(k+1)(k-2)}{k-1}\right)n$, i. e., $|E(J)| \geq \left(\frac{(k+1)(k-2)}{2(k-1)}\right)n$. ∎

6. Sharpness

First we prove Corollary 7, and then we will construct sparse 3-connected k-critical graphs. As it was pointed out in the introduction, Construction 58 and infinite series of 3-connected sparse 4- and 5-critical graphs are due to Toft [33] (based on [32]).

Proof of Corollary 7. By (1), if we construct an n_0-vertex k-critical graph for which our lower bound on $f_k(n_0)$ is exact, then the bound on $f_k(n)$ is exact for every n of the form $n_0 + s(k-1)$. So, by Corollary 5, we only need to construct

- a 5-critical 7-vertex graph with $\lceil 15\frac{1}{2} \rceil = 16$ edges,
- a 5-critical 8-vertex graph with $\lceil 17\frac{3}{4} \rceil = 18$ edges,
- a 6-critical 10-vertex graph with $\lceil 27\frac{1}{5} \rceil = 28$ edges,
- a 6-critical 12-vertex graph with $\lceil 32\frac{4}{5} \rceil = 33$ edges, and
- a 7-critical 14-vertex graph with $\lceil 45\frac{1}{3} \rceil = 46$ edges.

These graphs are presented in Figure 1. ∎

Construction 58 (Toft [33]). *Let G be a k-critical graph, $e = uv \in E(G)$, and $w \in V(G) - \{u,v\}$ be such that for all $(k-1)$-colorings ϕ of $G-e$, $\phi(w) = \phi(u) = \phi(v)$. Let $S_1 \cup S_2 \cup S_3$ be a partition of the vertex set X of a copy of K_{k-1} such that each S_i is non-empty. We define G' as follows: $V(G') = V(G) \cup V(X)$ and $E(G') = (E(G) - e) \cup E(X) \cup E'$, where*

$$E' = \{ua\colon a \in S_1\} \cup \{vb\colon b \in S_2\} \cup \{wc\colon c \in S_3\}.$$

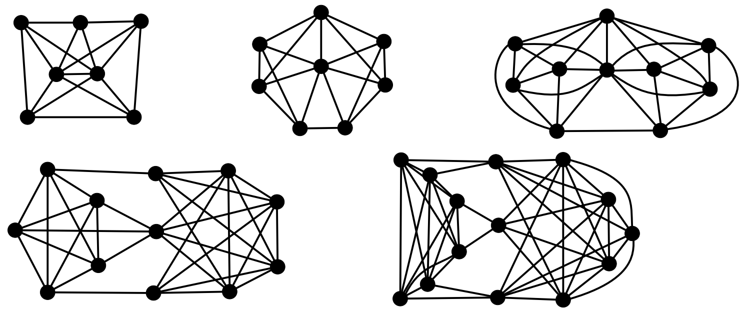

Figure 1. Minimal k-critical graphs

Claim 59. *If $k \geq 4$, G is a 3-connected k-critical graph and G' is created from G via Construction 58, then G' is a 3-connected k-critical graph.*

Proof. We will use the names and definitions from Construction 58.

If there exists a $(k-1)$-coloring ϕ of G', then all $k-1$ colors must appear on X. Then $\phi(u)$ appears on a vertex in S_2 or S_3. But then either $\phi(v) \neq \phi(u)$ or $\phi(w) \neq \phi(u)$, which contradicts the assumptions of Construction 58. So $\chi(G') \geq k$.

Suppose there exists an $f \in E(G')$ such that $\chi(G'-f) \geq k$. If $f \in E(G)$, then let ϕ_1 be a $(k-1)$-coloring of $G-f$. Because $e \in E(G)-f$, $\phi_1(u) \neq \phi_1(v)$, and so ϕ_1 extends easily to $G'-f$. If $f \subset X$, then a $(k-1)$-coloring of $G-e$ can be extended to $G'-f$, because X can be colored with $k-2$ colors, while $N(X) = \{u,v,w\}$ is colored with 1 color. If $f \in E'$, then a $(k-1)$-coloring of $G-e$ extends to $G'-f$, because the unique color on $\{u,v,w\}$ can be given to $f \cap X$. Therefore G' is k-critical.

Suppose now that there exists a set S such that $|S| < 3$ and there are nonempty A and B such that $E(A,B) = \emptyset$ and $A \cup B \cup S = V(G')$. Because k-critical graphs are 2-connected, $|S| = 2$. By Fact 12, S is independent. Because X is a clique, without loss of generality $X \subsetneq A \cup S$. By construction, $A - X \neq \emptyset$, so S also separates $G-e$. Since $\kappa(G) \geq 3$, $S \cap X = \emptyset$ and e has an endpoint in each component of $G-S-e$. So we may assume that $u \in A$ and $v \in B$. Since v has a neighbor in $S_2 \subset X$, this contradicts to $S \cap X = \emptyset$ and $X \subsetneq A \cup S$. ∎

The assumptions in Construction 58 are strong. Most edges e in k-critical graphs do not have such a vertex w, and some k-critical graphs do not have any edge-vertex pairs (e,w) that satisfy the assumptions. We will construct an infinite family $\mathbb{G}_k$ of sparse 3-connected graphs that do satisfy the assumptions.

The family is generated for each k by finding a small 3-connected k-critical graph G'_k such that $\rho_k(G'_k)=y_k$. We will describe a subgraph $H'_k \le G'_k$ with two vertices, u and w, such that in any $(k-1)$-coloring ϕ' of H'_k, $\phi'(u)=\phi'(w)$. Construction 58 can then be applied to G'_k, using any edge e incident to u that is not in H'_k and not incident to w. Because Construction 58 does not decrease the degree of u, this process can be iterated indefinitely to populate $\mathbb{G}_k$.

Note that Construction 58 adds the same number of vertices and edges as DHGO-composition with $G_2 = K_k$. Therefore, every graph $G \in \mathbb{G}_k$ has $\rho_k(G)=y_k$. Furthermore, G is also k-critical and 3-connected, and therefore not k-Ore. This implies the sharpness of Theorem 6.

All that is left is to find suitable graphs for G'_k and H'_k. Figure 2 illustrates G'_4 and G'_5. We will need a second construction for larger k.

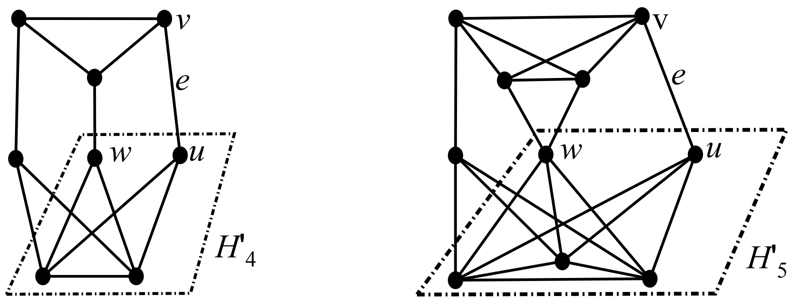

Figure 2. Graphs G'_4 and G'_5, with substructures labeled for constructing $\mathbb{G}_4$ and $\mathbb{G}_5$

Construction 60. *Fix a t such that $1 \le t < k/2$. Let*

$$V(H_{k,t}) = \{u_1, u_2, \ldots, u_{k-1}, v_1, v_2, \ldots, v_{k-1}, w\}$$

and

$$E(H_{k,t}) = \{u_iu_j\colon 1 \le i < j \le k-1\} \cup \{v_iv_j\colon 1 \le i < j \le k-1\} \cup \{u_iv_j\colon i,j \le t\} \cup \{wu_i\colon i > t\} \cup \{wv_i\colon i > t\}.$$

By construction, $H_{k,1}$ is a k-Ore graph, $H_{k,t}$ is k-critical, $\kappa(H_{k,t})=t+1$, $|V(H_{k,t})|=2k-1$, and $|E(H_{k,t})|=k(k-1)-2t+t^2$. Moreover, $\rho_k(H_{k,2})=y_k$. For $k \ge 6$, we choose $G'_k = H_{k,2}$. We will next find H'_k for $k \ge 6$, which will complete the argument.

Claim 61. *Let $H'_k = H_{k,2} - u_1v_1$. Then in every $(k-1)$-coloring ϕ' of H'_k, $\phi'(u_1)=\phi'(v_1)=\phi'(w)$.*

Proof. Let ϕ' be a $(k-1)$-coloring of H'_k. Note that each of the $(k-1)$ colors appears both, on $\{u_1,u_2,\dots,u_{k-1}\}$ and on $\{v_1,v_2,\dots,v_{k-1}\}$. Then $\phi'(w)$ appears on a vertex $a\in\{u_1,u_2\}$ and again on a vertex $b\in\{v_1,v_2\}$. So $ab\notin E(G)$, which implies that $a=u_1$ and $b=v_1$. ∎

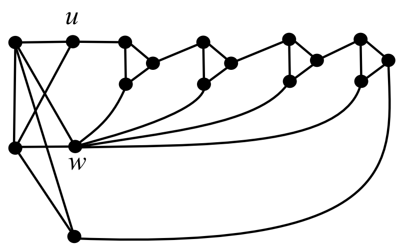

Figure 3. An example of a graph in $\mathbb{G}_4$

7. Algorithm

The proof of Theorem 4 was constructive, and provided an algorithm for $(k-1)$-coloring of sparse graphs. Let $P_k(G)$ be the minimum of $\rho_{k,G}(W)$ over all $W\subseteq V(G)$ with $2\le|W|$.

Theorem 62 ([26]). *If $k\ge 4$, then every n-vertex graph G with $P_k(G)> k(k-3)$ can be $(k-1)$-colored in $O(k^{3.5}n^{6.5}\log(n))$ time.*

We present below a polynomial-time algorithm for checking whether a given graph is a k-Ore graph. Together with an analog of the algorithm in Theorem 62 that uses the proof of Theorem 6 instead of Theorem 4, it would yield a polynomial-time algorithm that for every n-vertex graph G with $P_k(G)>y_k$ either finds a $(k-1)$-coloring of G or finds a subgraph of G that is a k-Ore graph.

Our algorithm to determine whether an n-vertex graph G is k-Ore is simple:

0. If G is K_k, return "yes."

1. Check whether $n\equiv 1 \pmod{k-1}$ and $|E(G)|=\frac{(k+1)(k-2)|V(G)|-k(k-3)}{2(k-1)}$. If not, then return "no."

2. Check whether the connectivity of G is exactly 2. If not, then return "no." Otherwise, choose a separating set $\{x,y\}$.

3. If $G-x-y$ has more than two components or $xy\in E(G)$, then return "no." Otherwise, let A and B be the vertex sets of the two components

of $G-x-y$. If $\{|A| \pmod{k-1}, |B| \pmod{k-1}\} \neq \{k-2, 0\}$, then return “no”. Otherwise, rename A and B so that $|A| \pmod{k-1} = k-2$ and $|B| \pmod{k-1} = 0$.

4. Create graphs $\widetilde{G}(x,y)$ and $\check{G}(x,y)$ as defined in Fact 14. Recurse on each of $\widetilde{G}(x,y)$ and $\check{G}(x,y)$. If at least one recursion call returns “no,” then return “no.” Otherwise, return “yes.”

The longest procedure in this algorithm is checking whether the connectivity of G is exactly 2 at Step 2, which has complexity $O(kn^3)$ because $|E(G)| \leq kn/2$. And it will be called fewer than $2n/(k-2)$ times. So the overall complexity is at most $O(n^4)$.

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