A SHARP DIRAC-ERDŐS TYPE BOUND FOR LARGE GRAPHS

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Abstract. Let $k \geq 3$ be an integer, $h_k(G)$ be the number of vertices of degree at least $2k$ in a graph G, and $\ell_k(G)$ be the number of vertices of degree at most $2k-2$ in G. Dirac and Erdős proved in 1963 that if $h_k(G) - \ell_k(G) \geq k^2 + 2k - 4$, then G contains k vertex-disjoint cycles. For each $k \geq 2$, they also showed an infinite sequence of graphs $G_k(n)$ with $h_k(G_k(n)) - \ell_k(G_k(n)) = 2k - 1$ such that $G_k(n)$ does not have k disjoint cycles. Recently, the authors proved that, for $k \geq 2$, a bound of $3k$ is sufficient to guarantee the existence of k disjoint cycles and presented for every k a graph $G_0(k)$ with $h_k(G_0(k)) - \ell_k(G_0(k)) = 3k-1$ and no k disjoint cycles. The goal of this paper is to refine and sharpen this result: We show that the Dirac–Erdős construction is optimal in the sense that for every $k \geq 2$, there are only finitely many graphs G with $h_k(G) - \ell_k(G) \geq 2k$ but no k disjoint cycles. In particular, every graph G with $|V(G)| \geq 19k$ and $h_k(G) - \ell_k(G) \geq 2k$ contains k disjoint cycles.

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1. Introduction

For a graph G, let $|G| = |V(G)|$, $\|G\| = |E(G)|$, and $\delta(G)$ be the minimum degree of a vertex in G. For a positive integer k and a graph G, define $H_k(G)$ to be the subset of vertices with degree at least $2k$ and $L_k(G)$ to be the subset of vertices of degree at most $2k-2$ in G. Two graphs are *disjoint* if they have no common vertices.

Every graph with minimum degree at least 2 contains a cycle. The following seminal result of Corrádi and Hajnal [2] generalizes this fact.

Theorem 1.1. [2] *Let G be a graph and k a positive integer. If $|G| \geq 3k$ and $\delta(G) \geq 2k$, then G contains k disjoint cycles.*

Both conditions in Theorem 1.1 are sharp. The condition $|G| \geq 3k$ is necessary as every cycle contains at least 3 vertices. Further, there are infinitely many graphs that satisfy $|G| \geq 3k$ and $\delta(G) = 2k-1$, but contain at most $k-1$ disjoint cycles. For example, for any $n \geq 3k$, let $G_n = K_n - E(K_{n-2k+1})$ where $K_{n-2k+1} \subseteq K_n$.

The Corrádi-Hajnal Theorem inspired several results related to the existence of disjoint cycles in a graph (e.g. [3, 4, 7, 5, 13, 11, 1, 12, 10, 9]). This paper focuses on the following theorem of Dirac and Erdős [3], one of the first attempts to generalize Theorem 1.1.

Theorem 1.2. [3] *Let $k \geq 3$ be an integer and G be a graph with $|H_k(G)| - |L_k(G)| \geq k^2 + 2k - 4$. Then G contains k disjoint cycles.*

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Dirac and Erdős suggested that the bound $k^2 + 2k - 4$ is not best possible and also constructed an infinite sequence of graphs $G_k(n)$ with $h_k(G_k(n)) - \ell_k(G_k(n)) = 2k - 1$ such that $G_k(n)$ does not have k disjoint cycles. They did not explicitly pose problems, and it seems that Erdős regretted not doing so, as later in [6] he remarked (about [3]): "This paper was perhaps undeservedly neglected; one reason was that we have few easily quotable theorems there, and do not state any unsolved problems." Here we consider questions that are implicit in [3].

For small graphs, the bound of $|H_k(G)| - |L_k(G)| \geq 2k$ is not sufficient to guarantee the existence of k disjoint cycles. Indeed, K_{3k-1} contains at most $k - 1$ disjoint cycles, so for small graphs, a bound of at least $3k$ is necessary. The authors [8] recently proved that $3k$ is also sufficient.

Theorem 1.3. [8] *Let $k \geq 2$ be an integer and G be a graph with $|H_k(G)| - |L_k(G)| \geq 3k$. Then G contains k disjoint cycles.*

There exist graphs G with at least $3k$ vertices and $|H_k(G)| - |L_k(G)| \geq 2k$ that do not contain k disjoint cycles. For example, consider the graph $G_0(k)$ obtained from K_{3k-1} by selecting a subset $S \subseteq V(K_{3k-1})$ with $|S| = k$, removing all edges in $G[S]$, adding an extra vertex x and the edges from x to each vertex in S. Then $|H_k(G_0(k))| - |L_k(G_0(k))| = 3k - 2$ and $|G_0(k)| = 3k$, but x is not in a triangle, so $G_0(k)$ contains at most $k - 1$ disjoint cycles.

In [8], the authors describe another graph $G_1(k)$, obtained from $G_0(k)$ by adding k vertices of degree 1, each adjacent to x. The graph $G_1(k)$ still contains only $k - 1$ disjoint cycles, but has $4k$ vertices and $|H_k(G_1(k))| - |L_k(G_1(k))| = 2k$. However, in the special case that G is planar, it is shown in [8] that the bound of $2k$ is sufficient.

Theorem 1.4. [8] *Let $k \geq 2$ be an integer and G be a planar graph. If*

$$|H_k(G)| - |L_k(G)| \geq 2k,$$

then G contains k disjoint cycles.

Further, when $k \geq 3$, a bound of $2k$ is also sufficient for graphs with no two disjoint triangles.

Theorem 1.5. [8] *Let $k \geq 3$ be an integer and G be a graph such that G does not contain two disjoint triangles. If*

$$|H_k(G)| - |L_k(G)| \geq 2k,$$

then G contains k disjoint cycles.

In general, the bound of $2k$ is the best we may hope for, as witnessed by $K_{n-2k+1,2k-1}$ for $n \geq 4k$. Further, the graph $G_1(k)$ described above shows that a difference of $2k$ is not sufficient when $|G|$ is small. In [8], we were not able to determine whether for each k there are only finitely many such examples. In order to attract attention to this problem and based on known examples, we raised the following question.

Question 1.6. [8] Is it true that every graph G with $|G| \geq 4k+1$ and $|H_k(G)| - |L_k(G)| \geq 2k$ has k disjoint cycles?

The goal of this paper is to confirm that indeed for every $k \geq 2$, there are only finitely many graphs G with $h_k(G) - \ell_k(G) \geq 2k$ but no k disjoint cycles. We do this by answering Question 1.6 for graphs with at least $19k$ vertices.

Theorem 1.7. *Let $k \geq 2$ be an integer and G be a graph with $|G| \geq 19k$ and*

$$|H_k(G)| - |L_k(G)| \geq 2k.$$

Then G contains k disjoint cycles.

The remainder of this paper is organized as follows. The next two sections outline notation and previous results that will be used in the proof of Theorem 1.7. We also introduce Theorem 3.4, which is a more technical version of Theorem 1.7. Theorem 3.4 is proved in Section 4. The proof builds on the techniques of Dirac and Erdős [3] and uses Theorem 1.3 as the base case for our induction.

2. Notation

We mostly use standard notation. For a graph G and $x \in V(G)$, $N_G(x)$ is the set of all vertices adjacent to x in G, and the *degree* of x, denoted $d_G(x)$, is $|N_G(x)|$. When the choice of G is clear, we simplify the notation to $N(x)$ and $d(x)$, respectively. The complement of a graph G is denoted by $\overline{G}$. For an edge $xy \in E(G)$, G/xy denotes the graph obtained from G by contracting xy; the new vertex is denoted by v_{xy}.

For disjoint sets $U, U' \subseteq V(G)$, we write $\|U, U'\|_G$ for the number of edges from U to U'. When the choice of G is clear, we will write $\|U, U'\|$ instead. If $U = \{u\}$, then we will write $\|u, U'\|$ instead of $\|\{u\}, U'\|$. The *join* $G \vee G'$ of two graphs is $G \cup G' \cup \{xx' : x \in V(G) \text{ and } x' \in V(G')\}$. Let SK_m denote the graph obtained by subdividing one edge of the complete m-vertex graph K_m.

Given an integer k, we say a vertex in $H_k(G)$ is *high*, and set $h_k(G) = |H_k(G)|$. A vertex in $L_k(G)$ is *low*. Set $\ell_k(G) = |L_k(G)|$. A vertex v is in $V^i(G)$ if $d_G(v) = i$. Similarly, $v \in V^{\leq i}(G)$ if $d_G(v) \leq i$ and $v \in V^{\geq i}(G)$ if $d_G(v) \geq i$. In these terms, $H_k(G) = V^{\geq 2k}(G)$ and $L_k(G) = V^{\leq 2k-2}(G)$.

We say that $x, y, z \in V(G)$ *form a triangle* $T = xyzx$ in G if $G[\{x, y, z\}]$ is a triangle. If $v \in \{x, y, z\}$, then we say $v \in T$. A *set $\mathcal{T}$ of disjoint triangles* is a set of subgraphs of G such that each subgraph is a triangle and all the triangles are disjoint. For a set $\mathcal{S}$ of graphs, let $\bigcup \mathcal{S} = \bigcup \{V(S) : S \in \mathcal{S}\}$. For a graph G, let $c(G)$ be the maximum number of disjoint cycles in G and $t(G)$ be the maximum number of disjoint triangles in G. When the graph G and integer k are clear from the context, we use H and L for $H_k(G)$ and $L_k(G)$, respectively. The sizes of H and L will be denoted by h and ℓ, respectively.

3. Preliminaries

As shown in [10], if a graph G with $|G| \geq 3k$ and $\delta(G) \geq 2k-1$ does not contain a large independent set, then with two exceptions, G contains k disjoint cycles:

Theorem 3.1. [10] *Let $k \geq 2$. Let G be a graph with $|G| \geq 3k$ and $\delta(G) \geq 2k-1$ such that G does not contain k disjoint cycles. Then*

(1) G contains an independent set of size at least $|G| - 2k + 1$, or
(2) k is odd and $G = 2K_k \vee \overline{K_k}$, or
(3) $k = 2$ and G is a wheel.

The theorem gives the following corollary.

Corollary 3.2. *Let $k \geq 2$ be an integer and G be a graph with $|G| \geq 3k$. If $h \geq 2k$ and $\delta(G) \geq 2k-1$ (i.e. $L = \emptyset$), then G contains k disjoint cycles.*

This corollary, along with the following theorem from [8] will be used in the proof.

Theorem 3.3. [8] *Let $k \geq 2$ be an integer and G be a graph such that $|G| \geq 3k$. If*

$$h - \ell \geq 2k + t(G),$$

then G contains k disjoint cycles.

We prove the following technical statement that implies Theorem 1.7, but is more amenable to induction.

Theorem 3.4. *Suppose $i, k \in \mathbb{Z}$, $k \geq i$ and $k \geq 2$. Let $\alpha = 16$ be a constant. If G is a graph with $|G| \geq \alpha k + 3i$ and $h \geq \ell + 3k - i$, then $c(G) \geq k$.*

Theorem 1.7 is the special case of Theorem 3.4 for $i = k$. The heart of this paper will be a proof of Theorem 3.4. In the remainder of this section we organize the induction and establish some preliminary results.

We argue by induction on i. The base case $i \leq 0$ follows from Theorem 1.3. Now suppose $i \geq 1$. The equations $|G| \geq h + \ell$ and $h - \ell \geq 2k$ give

$$\ell \leq \frac{|G|}{2} - k. \tag{1}$$

The *2-core* of a graph G is the largest subgraph $G' \subseteq G$ with $\delta(G') \geq 2$. It can be obtained from G by iterative deletion of vertices of degree at most 1. The following lemma was proved in [8].

Lemma 3.5. [8] *Suppose the 2-core of G contains at least 6 vertices and is not isomorphic to SK_5. If $h_2(G) - \ell_2(G) \geq 4$ then $c(G) \geq 2$.*

Now, we prove a result regarding minimal counterexamples to Theorem 3.4. Call a triangle T *good* if $T \cap L_k(G) \neq \emptyset$.

Lemma 3.6. *Suppose $i, k \in \mathbb{Z}$, $k \geq i$ and $k \geq 2$. Let $\alpha = 16$. If a graph G satisfies all of:*

(a) $|G| \geq \alpha k + 3i$,
(b) $h \geq \ell + 3k - i$,
(c) $c(G) < k$, and
(d) subject to (a–c), $\sigma := (k, i, |G| + \|G\|)$ is lexicographically minimum,

then all of the following hold:

(i) G has no isolated vertices;
(ii) $k \geq 3$;
(iii) $L(G) \cup V^{\geq 2k+1}(G)$ is independent;
(iv) if $x \in L(G)$, $d(x) \geq 2$, and $xy \in E$, then xy is in a triangle; and
(v) if $\mathcal{T}$ is a nonempty set of disjoint good triangles in G and $X := \bigcup \mathcal{T}$, then $\|v, X\| \geq 2|\mathcal{T}| + 1$ for at least two vertices $v \in V \setminus X$.

Proof. Assume (a–d) hold. Using Theorem 1.3, (a–c) imply $i \geq 1$; so the minimum in (d) is well defined. If (i) fails, then let v be an isolated vertex in G. Now $G' := G - v$ and $i' := i - 1$ satisfy conditions (a–c), contradicting (d). Hence, (i) holds.

For (ii), suppose $k = 2$. Then $t(G) \leq c(G) \leq 1$. If $i = 1$ then $h - \ell \geq 3k - i \geq 2k + t(G)$, so $c(G) \geq 2$ by Theorem 3.3. Thus $i = 2$ and $h - \ell = 4$. Using (1) and (i),

$$\|G\| \geq \frac{1}{2}(\ell + 3(|G| - \ell) + h) = \frac{1}{2}(3|G| + h - 2\ell)$$

$$= \frac{1}{2}(3|G| - \ell + 4) \geq \frac{1}{2}\left(3|G| - \left(\frac{|G|}{2} - 2\right) + 4\right)$$

$$= |G| + \frac{|G|}{4} + 3 \geq |G| + \frac{\alpha}{2} + \frac{3i}{4} + 3 = |G| + \frac{\alpha}{2} + \frac{9}{2}. \tag{2}$$

If G' is the 2-core of G, then $\|G'\| - |G'| \geq \|G\| - |G|$. Since $\alpha > 1$, (2) yields $\|G'\| > |G'| + 5$; so $|G'| > 5$ and $G' \ncong SK_5$. By Lemma 3.5, $c(G) \geq 2$, contradicting (c).

For (iii), suppose $e \in E(G[L \cup V^{\geq 2k+1}(G)])$, and set $G' := G - e$. Since G' is a spanning subgraph of G, it satisfies (a) and (c). Moreover, by the definition of G', $h_k(G') = h$ and $\ell_k(G') = \ell$, so (b) holds for G', which means (d) fails for G.

If (iv) fails, then let $G' = G/xy$ and $i' = i - 1$. Since $d_{G'}(v_{xy}) \geq d(y)$ and the degrees of all other vertices in G' are unchanged, G' and i' satisfy (a–c), contradicting (d).

Finally, suppose (v) fails, and let $u \in V \smallsetminus X$ with $\|u, X\|$ maximum. Then $\|v, X\| \leq 2|\mathcal{T}|$ for all $v \in V \smallsetminus (X + u)$. Set $G' = G - X$, $k' = k - |\mathcal{T}|$, and $i' = i - |\mathcal{T}| \leq k'$. Then $H \cap V(G') - u \subseteq H_{k'}(G')$ and $L_{k'}(G') - u \subseteq L \cap V(G')$. Since $\alpha \geq 3$, we have $|G'| \geq \alpha k' + 3i$; so G' satisfies (a). Let $\beta_1 = 1$ if $u \in H \smallsetminus H_{k'}(G')$; else $\beta_1 = 0$. Let $\beta_2 = 1$ if $u \in L_{k'}(G') \smallsetminus L$; else $\beta_2 = 0$. Then $\beta_1 + \beta_2 \leq |\mathcal{T}|$ and so

$$h_{k'}(G') \geq h - 2|\mathcal{T}| - \beta_1 \geq \ell + 3k - i - 2|\mathcal{T}| - \beta_1. \tag{3}$$

Since $\mathcal{T}$ is a set of good triangles, there are $|\mathcal{T}|$ in X that are low in G. Also, by assumption, there are at most $2|\mathcal{T}|$ vertices in $L_{k'}(G') - L_k(G)$. Hence, $\ell \geq \ell_{k'}(G') + |T| - \beta_2$, and combining with (3) yields

$$\begin{aligned} h_{k'}(G') &\geq (\ell_{k'}(G') + |\mathcal{T}| - \beta_2) + 3k - i - 2|\mathcal{T}| - \beta_1 \\ &\geq \ell_{k'}(G') - |\mathcal{T}| + 3(k' + |\mathcal{T}|) - (i' + |\mathcal{T}|) - \beta_1 - \beta_2 \\ &\geq \ell_{k'}(G') + 3k' - i'. \end{aligned}$$

This means G' satisfies (b). As $c(G') + |\mathcal{T}| \leq c(G) < k$, $c(G') < k'$. Thus G' satisfies (c). If $k' \geq 2$, then this contradicts the choice of k in (d), so (v) holds.

Otherwise, $k' = 1$, i.e., $|\mathcal{T}| = k - 1$ and so $|X| = 3k - 3$. Since each triangle in $\mathcal{T}$ has a low vertex, $|L \cap X| \geq |\mathcal{T}|$, and by (iii), $d_G(x) \leq 2k$ for each $x \in X$. Thus

$$\|X, V(G')\| < 2k|X| < 6k^2. \tag{4}$$

By (b), $|H \cap V(G')| - |L \cap V(G')| \geq 3k - i - |H \cap X| + |L \cap X| \geq 2k - i$. So,

$$\sum_{v \in V(G') \cap (H \cup L)} d_G(v) \geq 2k|H \cap V(G')| \geq 2k\frac{|V(G') \cap (H \cup L)| + (2k - i)}{2}.$$

By this and (4), we get

$$2\|G'\| = \sum_{v \in V(G')} d_G(v) - \|X, V(G')\| \geq k(|G'| + 2k - i) - \|X, V(G')\| \geq k(|G'| - 4k - i). \tag{5}$$

By (c), $c(G) \leq k - 1$, so G' has no cycle. Thus by (5),

$$2|G'| > 2\|G'\| \geq k(|G'| - 4k - i).$$

By (a), $|G'| \geq |G| - 3k \geq (\alpha-3)k + 3i = 13k + 3i$. Solving yields

$$k(4k+i) > (k-2)|G'| \geq (k-2)(13k+3i)$$
$$26k > 9k^2 + i(2k-6).$$

As $i \geq 0$, and $k \geq 3$ by (ii), this is a contradiction. □

4. Proof of Theorem 3.4

Fix k, i, and $G = (V, E)$ satisfying the hypotheses of Lemma 3.6. First choose a set $\mathcal{S}$ of disjoint good triangles with $s := |\mathcal{S}|$ maximum, and put $S = \bigcup \mathcal{S}$. Next choose a set $\mathcal{S}'$ of disjoint triangles, each contained in $V^{\leq 2k}(G) \smallsetminus S$, with $s' := |\mathcal{S}'|$ maximum, and put $S' = \bigcup \mathcal{S}'$. Say $\mathcal{S} = \{T_1, \dots, T_s\}$ and $\mathcal{S}' = \{T_{s+1}, \dots, T_{s+s'}\}$.

Let $\mathcal{H}$ be the directed graph defined on vertex set $\mathcal{S}$ by $CD \in E(\mathcal{H})$ if and only if there is $v \in C$ with $\|v, D\| = 3$. Here we allow graphs with no vertices. A vertex C' is *reachable* from a vertex C if $\mathcal{H}$ contains a directed CC'-path. In particular, each vertex C is reachable from itself via a CC-path of length 0.

Fact 4.1. *If $x \in L \smallsetminus S$ and $d(x) \geq 2$ then $N(x) \subseteq S$.*

Proof. Suppose $y \in N(x) \smallsetminus S$. As x is low, $x \notin S'$. By Lemma 3.6(iv), xy is in a triangle $xyzx$. As $\mathcal{S}$ is maximal, $z \in S$, so $z \in C$ for some $C \in \mathcal{S}$. Let

$$\mathcal{S}_0 = \{C' \in \mathcal{S} : C \text{ is reachable from } C' \text{ in } \mathcal{H}\}.$$

By Lemma 3.6(v), there is $w \in (V \smallsetminus \bigcup \mathcal{S}_0) - y$ with $\|w, \bigcup \mathcal{S}_0\| \geq 2|\mathcal{S}_0| + 1$. Then $\|w, D\| = 3$ for some $D \in \mathcal{S}_0$. By Lemma 3.6(iii), $w \neq x$. Further, $w \notin S$ as otherwise the triangle in $\mathcal{S}$ containing w is in $\mathcal{S}_0$, contradicting that $w \notin \bigcup \mathcal{S}_0$.

Let $D = C_1, \dots, C_j = C$ be a DC-path in $\mathcal{H}$, and for $i \in [j-1]$ let $x_i \in C_i$ with $\|x_i, C_{i+1}\| = 3$. Since each C_{i+1} contains a low vertex, by Lemma 3.6(iii), x_i is not a low vertex for each $i \in [j-1]$. Define new trianges $C_1' = C_1 - x_1 + w$, $C_j' = C_j - z + x_{j-1}$ and $C_i' = C_i - x_i + x_{i-1}$ for $i \in \{2, \dots, j-1\}$ and observe that each of these triangles contains a low vertex. Then, $\left(\mathcal{S} \smallsetminus \bigcup_{i=1}^{j} C_i\right) \cup \bigcup_{i=1}^{j} C_i' \cup \{xyzx\}$ is a set of $s+1$ disjoint good triangles. This contradicts the maximality of $\mathcal{S}$. □

Fact 4.2. *Each $v \in V$ is adjacent to at most 2 leaves. Moreover, if v is adjacent to 2 leaves, then $v \in V^{2k}$.*

Proof. Let v be adjacent to a leaf. By Lemma 3.6(iii), $v \in V^{2k-1} \cup V^{2k}$. Let X be the set of leaves adjacent to v, and put $G' = G - X$. Let $i' = i - (|X| - 1 - |\{v\} \cap V^{2k}|)$. Observe

$$\begin{aligned} h_k(G') - \ell_k(G') &\geq (h - |\{v\} \cap V^{2k}|) - (\ell + 1 - |X|) \\ &= h - \ell - |\{v\} \cap V^{2k}| + |X| - 1 \\ &\geq 3k - i - |\{v\} \cap V^{2k}| + |X| - 1 \\ &= 3k - i', \end{aligned}$$

so (b) holds for G', k and i'. Now, $|G'| \geq \alpha k + 3i - |X| = \alpha k + 3i' + 2|X| - 3(1 + |\{v\} \cap V^{2k}|)$. If $|X| \geq 3$, then $2|X| - 3(1 + |\{v\} \cap V^{2k}|) \geq 0$, so $|G'| \geq \alpha k + 3i'$ and (a) holds. As i' is at most i and $G' \subset G$, (d) does not hold for G, k, and i, a contradiction. Similarly, if

$v \in V^{2k-1}$ and $|X| = 2$, then $|G'| \geq \alpha k + 3i'$, so (a) still holds and G', k and i'. Thus this also contradicts (d) for G. $\square$

Let $G_1 = G - V^1$. Let $H^1 = V^{\geq 2k}(G_1)$, $R^1 = V^{2k-1}(G_1)$, $L^1 = L_k(G_1) \cap L$, and $M = L_k(G_1) \smallsetminus L^1$. Then $G_1 = G[H^1 \cup R^1 \cup M \cup L^1]$ and $V^{\geq 2k-1}(G) = H^1 \cup R^1 \cup M$. Since deleting a leaf does not decrease the difference $h - \ell$,

$$h_k(G_1) - \ell_k(G_1) \geq 3k - i. \tag{6}$$

Fact 4.3. *If $x \in M$, then x is in a triangle $xyzx$ in G with $d(x), d(y), d(z) \leq 2k$.*

Proof. Suppose $x \in M$. By Fact 4.2, either (i) $x \in V^{2k-1}$ and is adjacent to one leaf or (ii) $x \in V^{2k}$ and is adjacent to two leaves. Thus $d(x) \leq 2k$. We first claim:

$$x \text{ has a neighbor } y \text{ such that } 2 \leq d(y) \leq 2k. \tag{7}$$

Suppose not. Let X be the set consisting of x and the leaves adjacent to x. For each vertex $v \notin X$, $d_{G-X}(v) \geq d(v) - 1$, with equality if $v \in N(x)$. Moreover, if $v \in N(x)$, then $d_{G-X}(v) \geq 2k$. Therefore, $h_k(G - X) = h - |\{x\} \cap V^{2k}|$ and $\ell_k(G - X) = \ell - (|X| - 1)$. So

$$h_k(G - X) - \ell_k(G - X) = h - \ell + 1 \geq 3k - (i - 1)$$

and $|G - X| \geq |G| - 3 \geq \alpha k + 3(i - 1)$, contradicting the minimality of i. So (7) holds.

Now, suppose xy is not in a triangle. Let G' be formed from G by removing the leaves adjacent to x and contracting xy. By Fact 4.2, $|G'| \geq |G| - 3$. Since $d(x) \geq 2k - 1$ and x does not share neighbors with y, $d_{G'}(v_{xy}) \geq d(y)$. Similarly, $d_{G'}(v) = d(v)$ for all $v \in V(G') - v_{xy}$. Now, $h_k(G') - \ell_k(G') = h - \ell + 1 \geq 3k - (i - 1)$, contradicting the choice of i.

Let $xyzx$ be a triangle containing xy. If $d(z) \leq 2k$, we are done. Otherwise, let G'' be the graph obtained from G by removing the leaves adjacent to x and deleting the vertices x, y, and z. Observe $|G''| \geq |G| - 5 \geq \alpha(k - 1) + 3(i - 1)$. If there exists a vertex $u \in H \smallsetminus H_{k-1}(G'')$, then $N(u) \supseteq \{x, y, z\}$, and $d(u) \leq 2k$, since $d(z) \geq 2k + 1$. In this case $xyux$ is the desired triangle. Similarly, if $v \in L_{k-1}(G'') \smallsetminus L$, then $xyvx$ is the desired triangle. Thus $h - h_{k-1}(G'') \leq 2 + |\{x\} \cap V^{2k}|$ and $\ell - \ell_{k-1}(G'') \geq 1 + |\{x\} \cap V^{2k}|$. Now,

$$h_{k-1}(G'') - \ell_{k-1}(G'') \geq h - \ell - 1 \geq 3k - i - 1 = 3(k - 1) - (i - 2).$$

By the minimality of G, $c(G'') \geq k - 1$. Hence $c(G) \geq k$, a contradiction. We conclude that $xyzx$ is a triangle with $d(x), d(y), d(z) \leq 2k$. $\square$

Fact 4.4. $s + s' \geq 1$.

Proof. Suppose $s + s' = 0$. In this case, Fact 4.3 implies $M = \emptyset$: indeed, if $v \in M$, there exists a triangle $vuwv$ with $d(v), d(u), d(w) \leq 2k$, contradicting the choice of $\mathcal{S}'$. By Fact 4.1 and since $\mathcal{S} = \emptyset$, all vertices in L have degree at most 1. By Lemma 3.6(i), all vertices in L are leaves in G and $L^1 = \emptyset$.

Now, for every $x \in H - H_k(G_1)$, there is a leaf $y \in L - L_k(G_1)$ such that $xy \in E(G)$. Hence,

$$h_k(G_1) \geq h_k(G_1) - \ell_k(G_1) \geq h - \ell \geq 2k.$$

By (1) and since $\alpha \geq 4$, $|G_1| \geq |G| - \ell \geq |G|/2 + k \geq \alpha k/2 + k \geq 3k$. Finally, $L_k(G_1) = L^1 \cup M = \emptyset$, so Corollary 3.2 implies G_1 (and also G) contains k disjoint cycles. $\square$

Let $G_2 = G \setminus (L \setminus S)$. So $|G_2| = |G| - |L| + |S|$ and, using (1) and the assumption $|G| \geq \alpha k + 3i$, observe

$$|G_2| \geq \frac{\alpha+2}{2}k + \frac{3i}{2}. \tag{8}$$

Proof of Theorem 3.4. Put $s^* = \max\{1, s\}$. Let $\mathcal{S}^* = \{T_1, \ldots, T_{s^*}\}$; by Fact 4.4, T_{s^*} exists. Put $S^* = \bigcup \mathcal{S}^*$. Let $W = V(G_2) \setminus S^*$, $F = G[W]$ and $k' = k - s^*$. It suffices to prove $c(F) \geq k'$.

Case 1: $s^* = k-1$. Since $k \geq 3$, $s^* \geq 2$. Thus, $s = s^* = k-1$. By Fact 4.2, all vertices in M have degree $2k-2$ in F. Let $M' = M \cap W$ and $H' = H(G_2) \cap W$. Fact 4.1 implies that if $v \in W$, then $d_{G_1}(v) = d_{G_2}(v)$. Thus

$$H' = H^1 \cap W \text{ and } L(G_1) \cap W = L(G_2) \cap W.$$

Hence, by (6),

$$\begin{aligned} 2k \leq h(G_1) - \ell(G_1) &\leq (|H(G_1) \cap S| + |H'|) - (|L(G_1) \cap S| + |M \cap W| + |L^1 \setminus S|) \\ &= (|H(G_1) \cap S| - |L(G_1) \cap S|) + |H'| - |M'| - |L^1 \setminus S| \\ &\leq (k-1) + |H'| - |M'|. \end{aligned} \tag{9}$$

Here, the last inequality holds because S contains $s = k-1$ low vertices and at most $2s = 2k-2$ high vertices. Equation (9) implies $|H'| - |M'| \geq k+1$. Further, if W does not contain a cycle, then

$$\begin{aligned} \|W, S\|_{G_2} &\geq \sum_{v \in W} d_{G_2}(v) - 2(|W|-1) \\ &\geq ((2k-1)|W| + |H'| - |M'|) - 2(|W|-1) \\ &\geq ((2k-1)|W| + k + 1) - 2(|W|-1) \\ &\geq (2k-3)|W| + k + 3. \end{aligned} \tag{10}$$

On the other hand, every triangle in $\mathcal{S}$ contains a low vertex. This fact, together with Lemma 3.6(iii) implies,

$$\|W, S\|_{G_2} \leq \sum_{w \in S} (d_{G_2}(w) - 2) \leq (k-1)(6k-8). \tag{11}$$

Therefore, combining (10) and (11), $|W| \leq 3(k-1) - \frac{4}{2k-3}$. Since $|S| = 3(k-1)$ and $|G_2| = |S| + |W|$, this contradicts (8) when $\alpha \geq 10$.

Case 2: $s^* \leq k-2$. Consider a vertex v in $V^{\leq 2k'-2}(F)$. Since every vertex in F has degree at least $2k-2$ in G_2, v must be adjacent to at least $2s^*$ vertices in S^*. Further, every vertex in S^* is adjacent to at most $2k-2$ vertices outside of S^*. Therefore,

$$2s^*|V^{\leq 2k'-2}(F)| \leq \|V^{\leq 2k'-2}(F), S^*\| \leq 3s^*(2k-2), \tag{12}$$

and so

$$|V^{\leq 2k'-2}(F)| \leq 3k-3. \tag{13}$$

Similarly, if $u \in V^{2k'-1}(F)$, then u is adjacent to at least $2s^*-1$ vertices in S^*. Moreover, there are at most $3s^*(2k-2) - \|V^{\leq 2k'-2}(F), S^*\|$ edges from $V^{2k'-1}(F)$ to S^*. So,

$$(2s^*-1)|V^{2k'-1}(F)| \leq \|V^{2k'-1}(F), S^*\| \leq 3s^*(2k-2) - \|V^{\leq 2k'-2}(F), S^*\|,$$

and, combining with (12) gives,

$$\begin{aligned} |V^{2k'-1}(F)| &\leq \frac{2s^*(3k-3)}{2s^*-1} - \frac{2s^*|V^{\leq 2k'-2}(F)|}{2s^*-1} \\ &= 3k-3+\frac{3k-3}{2s^*-1} - \frac{2s^*|V^{\leq 2k'-2}(F)|}{2s^*-1}. \end{aligned} \tag{14}$$

Using (13) and (14), we see that

$$\begin{aligned} h_{k'}(F) - \ell_{k'}(F) &= |W| - 2|V^{\leq 2k'-2}(F)| - |V^{2k'-1}(F)| \\ &\geq |W| - 2|V^{\leq 2k'-2}(F)| - \left(3k-3+\frac{3k-3}{2s^*-1} - \frac{2s^*|V^{\leq 2k'-2}(F)|}{2s^*-1}\right) \\ &= |W| - \frac{(2s^*-2)|V^{\leq 2k'-2}(F)|}{2s^*-1} - 3k+3-\frac{3k-3}{2s^*-1} \\ &\geq |W| - \frac{(2s^*-2)(3k-3)}{2s^*-1} - 3k+3-\frac{3k-3}{2s^*-1} \\ &= |W| + \left(-(3k-3)+\frac{3k-3}{2s^*-1}\right) - 3k+3-\frac{3k-3}{2s^*-1} \\ &= |W| - 6k+6 \\ &\geq \left(\frac{\alpha+2}{2}k+\frac{3i}{2}-3s^*\right) - 6k+6 \\ &\geq \frac{\alpha+2}{2}k+\frac{3i}{2}-9k+6+3k'. \end{aligned}$$

When $\alpha \geq 16$, this is at least $3k'$. Further, $k' \geq 2$, since $s^* \leq k-2$. Therefore, Theorem 1.3 implies that F contains k' disjoint cycles. □

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