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Packing chromatic number of cubic graphs

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ABSTRACT

A *packing k-coloring* of a graph G is a partition of $V(G)$ into sets $V_1, \ldots, V_k$ such that for each $1 \leq i \leq k$ the distance between any two distinct $x, y \in V_i$ is at least $i + 1$. The *packing chromatic number*, $\chi_p(G)$, of a graph G is the minimum k such that G has a packing k-coloring. Sloper showed that there are 4-regular graphs with arbitrarily large packing chromatic number. The question whether the packing chromatic number of subcubic graphs is bounded appears in several papers. We answer this question in the negative. Moreover, we show that for every fixed k and $g \geq 2k + 2$, almost every n-vertex cubic graph of girth at least g has the packing chromatic number greater than k.

1. Introduction

For a positive integer i, a set S of vertices in a graph G is *i-independent* if the distance in G between any two distinct vertices of S is at least $i + 1$. In particular, a 1-independent set is simply an independent set.

A *packing k-coloring* of a graph G is a partition of $V(G)$ into sets $V_1, \ldots, V_k$ such that for each $1 \leq i \leq k$, the set V_i is i-independent. The *packing chromatic number*, $\chi_p(G)$, of a graph G, is the minimum k such that G has a packing k-coloring. The notion of packing k-coloring was introduced in 2008 by Goddard, Hedetniemi, Hedetniemi, Harris and Rall [15] (under the name *broadcast coloring*) motivated by frequency assignment problems in broadcast networks. The concept has attracted a considerable attention recently: there are more than 25 papers on the topic (see e.g. [1,5–12,14,21] and references in them). In particular, Fiala and Golovach [10] proved that finding the packing chromatic number of a graph is NP-hard even in the class of trees. Sloper [21] showed that there are graphs with maximum degree 4 and arbitrarily large packing chromatic number.

The question whether the packing chromatic number of all *subcubic* graphs (i.e., the graphs with maximum degree at most 3) is bounded by a constant was not resolved. For example, Brešar, Klavžar, Rall, and Wash [7] wrote: *'One of the intriguing problems related to the packing chromatic number is whether it is bounded by a constant in the class of all cubic graphs'*. It was proved in [7,17–19,21] that it is indeed bounded in some subclasses of subcubic graphs. On the other hand, Gastineau and Togni [14] constructed a cubic graph G with $\chi_p(G) = 13$, and asked whether there are cubic graphs with a larger packing chromatic number. Brešar, Klavžar, Rall, and Wash [8] answered this question in affirmative by constructing a cubic graph G' with $\chi_p(G') = 14$. The main result of this paper answers the question in full: Indeed, there are cubic graphs with arbitrarily large packing chromatic number. Moreover, we prove that 'many' cubic graphs have 'high' packing chromatic number:

Theorem 1. *For each fixed integer $k \geq 12$ and $g \geq 2k + 2$, almost every n-vertex cubic graph G of girth at least g satisfies $\chi_p(G) > k$.*

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The theorem will be proved in the language of the so-called *Configuration model*, $\mathcal{F}_3(n)$. We will discuss this concept and some important facts on it in the next section. In Section 3 we give upper bounds on the sizes c_i of maximum i-independent sets in almost all cubic n-vertex graphs of large girth. The original plan was to show that for a fixed k and large n, the sum $c_1+\cdots+c_k$ is less than n. But we were not able to prove it (and maybe this is not true). In Section 4, we give an upper bound on the size of the union of an 1-independent, a 2-independent, and a 4-independent sets which is less than $c_1+c_2+c_4$. This allows us to prove Theorem 1 in the last section.

2. Preliminaries

2.1. Notation

We mostly use standard notation. If G is a (multi)graph and $v, u \in V(G)$, then $E_G(v,u)$ denotes the set of all edges in G connecting v and u, $e_G(v,u) := |E_G(v,u)|$, and $\deg_G(v) := \sum_{u\in V(G)\setminus\{v\}} e_G(v,u)$. For $A \subseteq V(G)$, $G[A]$ denotes the sub(multi)graph of G induced by A. The independence number of G is denoted by $\alpha(G)$. For $k \in \mathbb{Z}_{>0}$, $[k]$ denotes the set $\{1,\ldots,k\}$.

2.2. The configuration model

The configuration model is due in different versions to Bender and Canfield [2] and Bollobás [3,4]. Our work is based on the version of Bollobás. Let V be the vertex set of the graph, we are going to associate a 3-element set to each vertex in V. Let n be an even positive integer. Let $V_n = [n]$ and consider the Cartesian product $W_n = V_n \times [3]$. A *configuration/pairing* (of order n and degree 3) is a partition of W_n into $3n/2$ pairs, i.e., a perfect matching of elements in W_n. There are

$$\frac{\binom{3n}{2}\cdot\binom{3n-2}{2}\cdot\ldots\cdot\binom{2}{2}}{(3n/2)!} = (3n-1)!!$$

such matchings. Let $\mathcal{F}_3(n)$ denote the collection of all $(3n-1)!!$ possible pairings on W_n. We project each pairing $F \in \mathcal{F}_3(n)$ to a multigraph $\pi(F)$ on the vertex set V_n by ignoring the second coordinate. Then $\pi(F)$ is a 3-regular multigraph (which may or may not contain loops and multi-edges). Let $\pi(\mathcal{F}_3(n)) = \{\pi(F) : F \in \mathcal{F}_3(n)\}$ be the set of 3-regular multigraphs on V_n. By definition,

$$\textit{each simple graph } G \in \pi(\mathcal{F}_3(n)) \textit{ corresponds to } (3!)^n \textit{ distinct pairings in } \mathcal{F}_3(n). \tag{1}$$

We will call the elements of V_n - *vertices*, and of W_n - *points*.

Definition 2. Let $\mathcal{G}_g(n)$ be the set of all cubic graphs with vertex set $V_n = [n]$ and girth at least g and $\mathcal{G}'_g(n) = \{F \in \mathcal{F}_3(n) : \pi(F) \in \mathcal{G}_g(n)\}$.

We will use the following result:

Theorem 3 (*Wormald [22], Bollobás [3]*). *For each fixed $g \geq 3$,*

$$\lim_{n\to\infty} \frac{|\mathcal{G}'_g(n)|}{|\mathcal{F}_3(n)|} = \exp\left\{-\sum_{k=1}^{g-1}\frac{2^{k-1}}{k}\right\}. \tag{2}$$

Remark. When we say that *a pairing F has a multigraph property* $\mathcal{A}$, we mean that $\pi(F)$ has property $\mathcal{A}$.

Since dealing with pairings is simpler than working with labeled simple regular graphs, we need the following well-known consequence of Theorem 3.

Corollary 4 (*[20](Corollary 1.1), [16](Theorem 9.5)*). *For fixed $g \geq 3$, any property that holds for $\pi(F)$ for almost all pairings $F \in \mathcal{F}_3(n)$ also holds for almost all graphs in $\mathcal{G}_g(n)$.*

Proof. Suppose property $\mathcal{A}$ holds for $\pi(F)$ for almost all $F \in \mathcal{F}_3(n)$. Let $\mathcal{H}(n)$ denote the set of graphs in $\mathcal{G}_g(n)$ that do *not* have property $\mathcal{A}$ and $\mathcal{H}'(n) = \{F \in \mathcal{F}_3(n) : \pi(F) \in \mathcal{H}(n)\}$. Let $\mathcal{B}(n)$ denote the set of pairings $F \in \mathcal{F}_3(n)$ such that $\pi(F)$ does not have property $\mathcal{A}$. Then $\mathcal{H}'(n) \subseteq \mathcal{B}(n)$. Hence by the choice of $\mathcal{A}$,

$$\frac{|\mathcal{H}'(n)|}{|\mathcal{F}_3(n)|} \leq \frac{|\mathcal{B}(n)|}{|\mathcal{F}_3(n)|} \to 0 \qquad \text{as } n\to\infty. \tag{3}$$

By (1), we have

$$\frac{|\mathcal{H}(n)|}{|\mathcal{G}_g(n)|} = \frac{|\mathcal{H}(n)|}{|\mathcal{H}'(n)|}\cdot\frac{|\mathcal{H}'(n)|}{|\mathcal{G}'_g(n)|}\cdot\frac{|\mathcal{G}'_g(n)|}{|\mathcal{G}_g(n)|} = \frac{1}{(3!)^n}\cdot\frac{|\mathcal{H}'(n)|}{|\mathcal{G}'_g(n)|}\cdot(3!)^n = \frac{|\mathcal{H}'(n)|}{|\mathcal{G}'_g(n)|}.$$

Furthermore,

$$\frac{|\mathcal{H}'(n)|}{|\mathcal{G}'_g(n)|} = \frac{|\mathcal{H}'(n)|}{|\mathcal{F}_3(n)|} \cdot \frac{|\mathcal{F}_3(n)|}{|\mathcal{G}'_g(n)|}. \tag{4}$$

By (3) and Theorem 3, the right-hand side of (4) tends to 0 as n tends to infinity. ■

3. Bounds for $c_1, c_2, \ldots$

We will use the following theorem of McKay [20].

Theorem 5 (*McKay [20]*)**.** *For every $\varepsilon > 0$, there exists an $N > 0$ such that for each $n > N$,*

$$|\{F \in \mathcal{F}_3(n) : c_1(\pi(F)) > 0.45537n\}| < \varepsilon \cdot (3n-1)!!.$$

Definition 6. A 3-*regular tree* is a tree such that each vertex has degree 3 or 1. A $(3, k, a)$-*tree* is a rooted 3-regular tree T with root a of degree 3 such that the distance in T from each of the leaves to a is k (see Fig. 1).

Definition 7. For a positive integer s and a vertex a in a graph G, the *ball $B_G(a, s)$ in G of radius s with center a* is $\{v \in V(G) : d_G(v, a) \le s\}$, where $d_G(v, a)$ denotes the distance in G from v to a.

We first prove simple bounds on $c_{2k}(G)$ and $c_{2k+1}(G)$ when $G \in \mathcal{G}_{2k+2}(n)$.

Lemma 8. *Let j be a fixed positive integer and $n > g \ge 2j + 2$. Then for every $G \in \mathcal{G}_g(n)$,*

$$\text{(i)} \quad c_{2j}(G) \le \frac{n}{3 \cdot 2^j - 2},$$

and

$$\text{(ii)} \quad c_{2j+1}(G) \le \frac{c_1(G)}{2^{j+1} - 1}.$$

Proof. (i) Let C_{2j} be a $2j$-independent set in G with $|C_{2j}| = c_{2j}(G)$. Since the distance between any distinct $a, b \in C_{2j}$ is at least $2j+1$, the balls $B_G(a, j)$ for all distinct $a \in C_{2j}$ are disjoint. Moreover, since $g \ge 2j+2$, each ball $B_G(a, j)$ induces a $(3, j, a)$-tree T_a, and hence has

$$1 + 3 + 3 \cdot 2 + 3 \cdot 2^2 + \cdots + 3 \cdot 2^{j-1} = 3 \cdot 2^j - 2$$

vertices. This proves (i).

(ii) Let C_{2j+1} be a $(2j + 1)$-independent set in G with $|C_{2j+1}| = c_{2j+1}(G)$. As in the proof of (i), the balls $B_G(a, j)$ for distinct $a \in C_{2j}$ are disjoint, and each $B_G(a, j)$ induces a $(3, j, a)$-tree T_a. But in this case, in addition, the balls with centers in distinct vertices of C_{2j+1} are at distance at least 2 from each other. Let S_i be the set of vertices in T_a at distance i from a. Then $|S_0| = 1$, and for each $1 \le i \le j$, $|S_i| = 3 \cdot 2^{i-1}$. If follows that the set $I_a = \bigcup_{i=0}^{\lfloor j/2 \rfloor} S_{j-2i}$ is independent, and

$$|I_a| = \sum_{i=0}^{\lfloor j/2 \rfloor} |S_{j-2i}| = 2^{j+1} - 1.$$

Therefore $I := \bigcup_{a \in C_{2j+1}} I_a$ is an independent set in G and $|I| = (2^{j+1} - 1)c_{2j+1}(G)$. This implies (ii). ■

Lemma 9. *Let k be a fixed positive integer and x be a real number with $0 < x < \frac{1}{3 \cdot 2^k - 2}$. The number of pairings $F \in \mathcal{G}'_{2k+2}(n)$ such that $\pi(F)$ has a $2k$-independent vertex set of size xn is at most*

$$q(n, k, x) := \binom{n}{xn} \cdot (3n - (6 \cdot 2^k - 6)xn - 1)!! \cdot \prod_{i=0}^{k-1} \binom{(1 - (3 \cdot 2^i - 2)x)n}{3 \cdot 2^i xn} \cdot (3 \cdot 2^i xn)! \cdot 3^{3 \cdot 2^i xn}.$$

Proof. To prove the lemma, we will show that the total number of $2k$-independent sets of size xn in $\pi(F)$ over all $F \in \mathcal{G}'_{2k+2}(n)$ does not exceed $q(n, k, x)$. Below we describe a procedure of constructing for every $C \subset [n]$ with $|C| = xn$ all pairings $F \in \mathcal{G}_{2k+2}(n)$ for which C is $2k$-independent in $\pi(F)$. Not every obtained pairing will be in $\mathcal{G}'_{2k+2}(n)$, but every $F \in \mathcal{G}'_{2k+2}(n)$ such that C is a $2k$-independent set in $\pi(F)$ will be a result of this procedure:

1. We choose a vertex set C of size xn from $[n]$. There are $\binom{n}{xn}$ ways to do it.

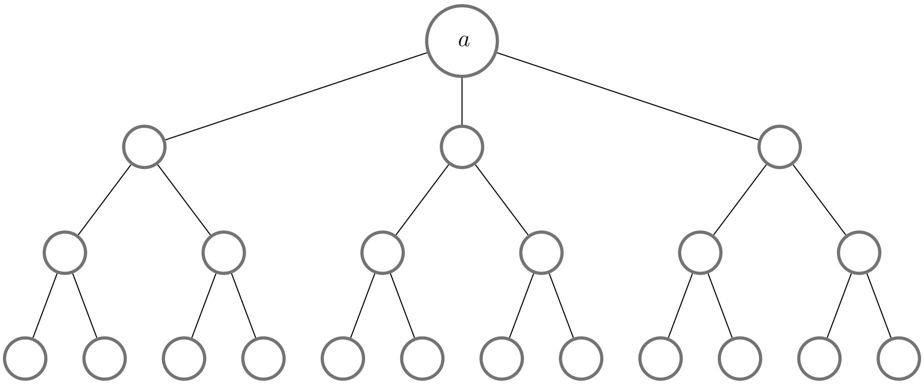

Fig. 1. A $(3, 3, a)$-tree T_a.

2. In order C to be $2k$-independent and $\pi(F)$ to have girth at least $2k+2$, all the balls of radius k with the centers in C must be disjoint, and for each $a \in C$, the ball $B_{\pi(F)}(a, k)$ must induce a $(3, k, a)$-tree. Thus, we have $\binom{(1-x)n}{3xn}$ ways to choose the neighbors of C, call it $N(C)$, $(3xn)!$ ways to determine which vertex in $N(C)$ will be the neighbor for each point in $\pi^{-1}(C)$, and 3^{3xn} ways to decide which point of each vertex in $N(C)$ is adjacent to the corresponding point in $\pi^{-1}(C)$. Each vertex of $N(C)$ will have 2 free points left at this moment, and in total the set $\pi^{-1}(N(C))$ has now $2 \cdot 3xn = 6xn$ free points.
3. Similarly to the previous step, consecutively for $i = 1, 2, \dots, k-1$, we will decide which vertices and points are in the set $\pi^{-1}(N^{i+1}(C))$ of the vertices at distance i from C, as follows. Before the ith iteration, we have $3x \cdot 2^i n$ free points in the $3x \cdot 2^{i-1} n$ vertices of $\pi^{-1}(N^i(C))$, and

$$|C \cup N^1(C) \cup \cdots \cup N^i(C)| = xn\left(1 + 3(1 + 2 + \cdots + 2^{i-1})\right) = (3 \cdot 2^i - 2)xn.$$

We choose $3x \cdot 2^i n$ vertices out of the remaining $\left(1 - (3 \cdot 2^i - 2)x\right) n$ vertices to include into $N^{i+1}(C)$, then we have $(3x \cdot 2^i n)!$ ways to determine which vertex in $N^{i+1}(C)$ will be the neighbor for each free point in $\pi^{-1}(N^i(C))$, and $3^{3x \cdot 2^i n}$ ways to decide which point of each vertex in $N^{i+1}(C)$ is adjacent to the corresponding point in $\pi^{-1}(N^i(C))$.
4. Finally, there are $3n - (6 \cdot 2^k - 6)xn$ free points left and we have $(3n - (6 \cdot 2^k - 6)xn - 1)!!$ ways to pair them.

Multiplying the quantities in 1–4 above, we obtain $q(n, k, x)$. This proves the bound. ■

In the proofs below we will use Stirling's formula: For every $n \geq 1$,

$$\sqrt{2\pi n}\left(\frac{n}{e}\right)^n \leq n! \leq \sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^{1/12n}. \tag{5}$$

Corollary 10. *Let $g \geq 22$ be fixed. For every $\varepsilon > 0$, there exists an $N > 0$ such that for each $n > N$,*

$$|\{G \in \mathcal{G}_g(n) \, : \, c_2(G) > 0.236n =: b_2 n\}| \, < \varepsilon \cdot |\mathcal{G}_g(n)|, \tag{6}$$

$$|\{G \in \mathcal{G}_g(n) \, : \, c_4(G) > 0.082n =: b_4 n\}| \, < \varepsilon \cdot |\mathcal{G}_g(n)|, \tag{7}$$

$$|\{G \in \mathcal{G}_g(n) \, : \, c_6(G) > 0.03n =: b_6 n\}| \, < \varepsilon \cdot |\mathcal{G}_g(n)|, \tag{8}$$

$$|\{G \in \mathcal{G}_g(n) \, : \, c_8(G) > 0.011n =: b_8 n\}| \, < \varepsilon \cdot |\mathcal{G}_g(n)|, \tag{9}$$

and

$$|\{G \in \mathcal{G}_g(n) \, : \, c_{10}(G) > 0.004n =: b_{10} n\}| \, < \varepsilon \cdot |\mathcal{G}_g(n)|. \tag{10}$$

Proof. By Lemma 9,

$$\begin{aligned} q(n,k,x) &= \binom{n}{xn}\cdot((3n-(6\cdot 2^k-6)xn-1)!!)\prod_{i=0}^{k-1}\binom{(1-(3\cdot 2^i-2)x)n}{3\cdot 2^i xn}\cdot((3\cdot 2^i xn)!)(3^{3\cdot 2^i xn}) \\ &= \frac{(3n-(6\cdot 2^k-6)xn-1)!!\cdot n!}{(xn)!\cdot((1-x)n)!}\cdot 3^{3xn+6xn+\cdots+3\cdot 2^{k-1}xn} \\ &\quad \cdot\frac{((1-x)n)!\cdot(3xn)!}{(3xn)!\cdot((1-4x)n)!}\cdot\frac{((1-4x)n)!\cdot(6xn)!}{(6xn)!\cdot((1-10x)n)!}\cdot\cdots\cdot\frac{((1-(3\cdot 2^{k-1}-2)x)n)!\cdot(3\cdot 2^{k-1}xn)!}{(3\cdot 2^{k-1}xn)!\cdot((1-(3\cdot 2^k-2)x)n)!} \\ &= \frac{(3n-(6\cdot 2^k-6)xn-1)!!\cdot n!}{(xn)!\cdot((1-(3\cdot 2^k-2)x)n)!}\cdot 3^{(3\cdot 2^k-3)xn}. \end{aligned}$$

We know that

$$(3n-1)!! = \frac{(3n)!!}{3n} \geq \frac{\sqrt{(3n)!}}{3n}$$

and

$$(3n-(6\cdot 2^k-6)xn-1)!! \leq \sqrt{(3n-(6\cdot 2^k-6)xn)!}.$$

Therefore,

$$\frac{q(n,k,x)}{(3n-1)!!} \leq (3n)\cdot\left(\frac{(3n-(6\cdot 2^k-6)xn)!}{(3n)!}\right)^{\frac{1}{2}}\cdot\frac{n!}{(xn)!\cdot((1-(3\cdot 2^k-2)x)n)!}\cdot 3^{(3\cdot 2^k-3)xn}.$$

Using Stirling's formula (5), we have

$$\begin{aligned} \frac{q(n,k,x)}{(3n-1)!!} &= O(n^2)\cdot\frac{\left(\frac{n}{e}\right)^{\frac{1}{2}\cdot(3n-(6\cdot 2^k-6)xn)}\cdot\left(\frac{n}{e}\right)^n}{\left(\frac{n}{e}\right)^{\frac{3n}{2}}\cdot\left(\frac{n}{e}\right)^{xn}\cdot\left(\frac{n}{e}\right)^{(1-(3\cdot 2^k-2)x)n}}\cdot\left(\frac{(1-(2^{k+1}-2)x)^{1.5-(3\cdot 2^k-3)x}}{x^x(1-(3\cdot 2^k-2)x)^{1-(3\cdot 2^k-2)x}}\right)^n \\ &= O(n^2)\cdot\left(\frac{(1-(2^{k+1}-2)x)^{1.5-(3\cdot 2^k-3)x}}{x^x(1-(3\cdot 2^k-2)x)^{1-(3\cdot 2^k-2)x}}\right)^n. \end{aligned}$$

Let

$$f(x,k) = \frac{(1-(2^{k+1}-2)x)^{1.5-(3\cdot 2^k-3)x}}{x^x(1-(3\cdot 2^k-2)x)^{1-(3\cdot 2^k-2)x}}, \tag{11}$$

so that

$$\frac{q(n,k,x)}{|\mathcal{F}_3(n)|} = \frac{q(n,k,x)}{(3n-1)!!} = O(n^2)(f(x,k))^n. \tag{12}$$

By plugging $x = 0.236$ and $k = 1$ into (11) (using a computer or a good calculator), we see that $0 < f(0.236, 1) < 0.9964$. Since $f(x, 1)$ is a smooth function for $0 < x < 1$, there exists δ_1 such that $f(x, 1) < 0.9964$ for all $x \in [0.236 - \delta_1, 0.236]$. If $n > 1/\delta_1$, then there exists an $x_1 = x_1(n) \in [0.236 - \delta_1, 0.236]$ such that $x_1 n$ is an integer. By (12),

$$\frac{q(n,1,x_1 n)}{|\mathcal{F}_3(n)|} = O(n^2)(0.9964)^n \to 0 \qquad \text{as } n \to \infty.$$

By the definition of $q(n, k, x)$, (2) and Corollary 4, this implies (6).

Similarly, by plugging the corresponding values of x and k into (11), one can check that $0 < f(0.082, 2) < 0.9977$, $0 < f(0.03, 3) < 0.9981$, $0 < f(0.011, 4) < 0.996$, and $0 < f(0.004, 5) < 0.995$. Thus repeating the argument of the previous paragraph, we obtain that (7), (8), (9), (10) also hold. ■

Lemma 11. *Let k be a fixed positive integer and $0 < x < \frac{0.45537}{2^{k+1}-1}$. The number of pairings $F \in \mathcal{G}'_{2k+2}(n)$ such that $\pi(F)$ has a $(2k+1)$-independent vertex set of size xn is at most*

$$\begin{aligned} r(n,k,x) &:= \frac{\binom{n}{xn}\cdot(3(n-(3\cdot 2^k-2)xn))!\cdot(3(n-(4\cdot 2^k-2)xn)-1)!!}{(3(n-(4\cdot 2^k-2)xn))!} \\ &\quad \times\prod_{i=0}^{k-1}\binom{(1-(3\cdot 2^i-2)x)n}{3\cdot 2^i xn}\cdot(3\cdot 2^i xn)!\cdot 3^{3\cdot 2^i xn}. \end{aligned} \tag{13}$$

Proof. We will show that the total number of $(2k+1)$-independent sets of size xn in $\pi(F)$ over all $F \in \mathcal{G}'_{2k+2}(n)$ does not exceed $r(n, k, x)$. Below we describe a procedure of constructing for every set C of size xn in $[n]$ all pairings in $\mathcal{G}'_{2k+2}(n)$ for which C is $(2k+1)$-independent. Not every obtained pairing will be in $\mathcal{G}'_{2k+2}(n)$, but every $F \in \mathcal{G}'_{2k+2}(n)$ such that C is a $(2k+1)$-independent set in $\pi(F)$ will be a result of this procedure:

1. We choose a vertex set C of size xn from $[n]$. There are $\binom{n}{xn}$ ways to do it.
2. In order C to be $(2k+1)$-independent and $\pi(F)$ to have girth at least $2k+2$, all the balls of radius k with the centers in C must be disjoint, and for each $a \in C$, the ball $B_{\pi(F)}(a,k)$ must induce a $(3,k,a)$-tree. Thus, we have $\binom{(1-x)n}{3xn}$ ways to choose the neighbors of C, call it $N(C)$, $(3xn)!$ ways to determine which vertex in $N(C)$ will be the neighbor for each point in $\pi^{-1}(C)$, and 3^{3xn} ways to decide which point of each vertex in $N(C)$ is adjacent to the corresponding point in $\pi^{-1}(C)$. Each vertex of $N(C)$ will have 2 free points left at this moment, and in total the set $\pi^{-1}(N(C))$ has now $2\cdot 3xn = 6xn$ free points.
3. Similarly to the previous step, consecutively for $i = 1, 2, \ldots, k-1$, we will decide which vertices and points are in the set $\pi^{-1}(N^{i+1}(C))$ of the vertices at distance i from C, as follows. Before the ith iteration, we have $3x\cdot 2^i n$ free points in the $3x\cdot 2^{i-1}n$ vertices of $\pi^{-1}(N^i(C))$, and
$$|C \cup N^1(C) \cup \cdots \cup N^i(C)| = xn\left(1 + 3(1+2+\cdots+2^{i-1})\right) = (3\cdot 2^i - 2)xn.$$
We choose $3x\cdot 2^i n$ vertices out of the remaining $\left(1-(3\cdot 2^i-2)x\right)n$ vertices to include into $N^{i+1}(C)$, then we have $(3x\cdot 2^i n)!$ ways to determine which vertex in $N^{i+1}(C)$ will be the neighbor for each free point in $\pi^{-1}(N^i(C))$, and $3^{3x\cdot 2^i n}$ ways to decide which point of each vertex in $N^{i+1}(C)$ is adjacent to the corresponding point in $\pi^{-1}(N^i(C))$.
4. Let $N^0(C) := C$ and $S := \cup_{i=0}^{k} N^i(C)$. In order the distance between each pair of vertices in C to be at least $2k+2$, $N^k(C)$ has to be an independent set. Therefore, each of the $3x\cdot 2^k n$ free points in the $3x\cdot 2^{k-1}n$ vertices of $\pi^{-1}(N^k(C))$ has to be paired with one of the remaining $3(n-(3\cdot 2^k-2)xn)$ free points of $\pi^{-1}([n]-S)$ and we have
$$\frac{(3(n-(3\cdot 2^k-2)xn))!}{(3(n-(4\cdot 2^k-2)xn))!}$$
ways to do that.
5. Finally, there are $3(n-(4\cdot 2^k-2)xn)$ free points left and we have $(3(n-(4\cdot 2^k-2)xn)-1)!!$ ways to pair them.

The product of the numbers of choices in the above Steps 1–5 equals $r(n,k,x)$, which proves the lemma. ∎

Corollary 12. *Let $g \geq 24$ be fixed. For every $\varepsilon > 0$, there exists an $N > 0$ such that for each $n > N$,*

$$|\{G \in \mathcal{G}_g(n) \,:\, c_3(G) > 0.1394n =: b_3 n\}| < \varepsilon \cdot |\mathcal{G}_g(n)|, \tag{14}$$

$$|\{G \in \mathcal{G}_g(n) \,:\, c_5(G) > 0.05n =: b_5 n\}| < \varepsilon \cdot |\mathcal{G}_g(n)|, \tag{15}$$

$$|\{G \in \mathcal{G}_g(n) \,:\, c_7(G) > 0.0182n =: b_7 n\}| < \varepsilon \cdot |\mathcal{G}_g(n)|, \tag{16}$$

$$|\{G \in \mathcal{G}_g(n) \,:\, c_9(G) > 0.0063n =: b_9 n\}| < \varepsilon \cdot |\mathcal{G}_g(n)|, \tag{17}$$

and

$$|\{G \in \mathcal{G}_g(n) \,:\, c_{11}(G) > 0.0022n =: b_{11} n\}| < \varepsilon \cdot |\mathcal{G}_g(n)|. \tag{18}$$

Proof. By Lemma 11,

$$r(n,k,x) = \frac{\binom{n}{xn}\cdot(3(n-(3\cdot 2^k-2)xn))!\cdot(3(n-(4\cdot 2^k-2)xn)-1)!!}{(3(n-(4\cdot 2^k-2)xn))!} \times \prod_{i=0}^{k-1}\binom{(1-(3\cdot 2^i-2)x)n}{3\cdot 2^i xn}\cdot(3\cdot 2^i xn)!\cdot 3^{3\cdot 2^i xn} \tag{19}$$

$$= \frac{(3(n-(3\cdot 2^k-2)xn))!\cdot(3(n-(4\cdot 2^k-2)xn)-1)!!}{(3(n-(4\cdot 2^k-2)xn))!}\cdot\frac{n!}{(xn)!\cdot((1-x)n)!}\cdot 3^{3xn+6xn+\cdots+3\cdot 2^{k-1}xn}$$

$$\cdot\frac{((1-x)n)!\cdot(3xn)!}{(3xn)!\cdot((1-4x)n)!}\cdot\frac{((1-4x)n)!\cdot(6xn)!}{(6xn)!\cdot((1-10x)n)!}\cdot\cdots\cdot\frac{((1-(3\cdot 2^{k-1}-2)x)n)!\cdot(3\cdot 2^{k-1}xn)!}{(3\cdot 2^{k-1}xn)!\cdot((1-(3\cdot 2^k-2)x)n)!}$$

$$= \frac{(3(n-(3\cdot 2^k-2)xn))!\cdot(3(n-(4\cdot 2^k-2)xn)-1)!!}{(3(n-(4\cdot 2^k-2)xn))!}\cdot\frac{n!}{(xn)!\cdot((1-(3\cdot 2^k-2)x)n)!}\cdot 3^{(3\cdot 2^k-3)xn}.$$

By the definition of the double factorial,

$$(3n-1)!! \geq \frac{(3n)!!}{3n} \geq \frac{\sqrt{(3n)!}}{3n}$$

and

$$(3(n-(4\cdot 2^k-2)xn)-1)!! \leq \sqrt{(3(n-(4\cdot 2^k-2)xn))!}.$$

Therefore,

$$\frac{r(n,k,x)}{(3n-1)!!} \leq (3n)\cdot\left(\frac{(3(n-(4\cdot 2^k-2)xn))!}{(3n)!}\right)^{\frac{1}{2}}\cdot\frac{(3(n-(3\cdot 2^k-2)xn))!}{(3(n-(4\cdot 2^k-2)xn))!}$$
$$\cdot\frac{n!}{(xn)!\cdot((1-(3\cdot 2^k-2)x)n)!}\cdot 3^{(3\cdot 2^k-3)xn}.$$

By Stirling's formula (5),

$$\frac{r(n,k,x)}{(3n-1)!!} = O(n^3)\cdot\frac{\left(\frac{n}{e}\right)^{\frac{3}{2}\cdot(n-(4\cdot 2^k-2)xn)}\cdot\left(\frac{n}{e}\right)^{3(n-(3\cdot 2^k-2)xn)}\cdot\left(\frac{n}{e}\right)^{n}}{\left(\frac{n}{e}\right)^{\frac{3n}{2}}\cdot\left(\frac{n}{e}\right)^{3(n-(4\cdot 2^k-2)xn)}\cdot\left(\frac{n}{e}\right)^{xn}\cdot\left(\frac{n}{e}\right)^{(1-(3\cdot 2^k-2)x)n}}$$
$$\cdot\left(\frac{(1-(3\cdot 2^k-2)x)^{2-(6\cdot 2^k-4)x}}{x^x(1-(4\cdot 2^k-2)x)^{1.5-(6\cdot 2^k-3)x}}\right)^n$$
$$= O(n^3)\cdot\left(\frac{(1-(3\cdot 2^k-2)x)^{2-(6\cdot 2^k-4)x}}{x^x(1-(4\cdot 2^k-2)x)^{1.5-(6\cdot 2^k-3)x}}\right)^n.$$

Let

$$h(x,k) = \frac{(1-(3\cdot 2^k-2)x)^{2-(6\cdot 2^k-4)x}}{x^x(1-(4\cdot 2^k-2)x)^{1.5-(6\cdot 2^k-3)x}}, \tag{20}$$

so that

$$\frac{r(n,k,x)}{|\mathcal{F}_3(n)|} = \frac{r(n,k,x)}{(3n-1)!!} = O(n^3)(h(x,k))^n. \tag{21}$$

By plugging $x = 0.1394$ and $k = 1$ into (20) (using a computer or a calculator), we see that $0 < h(0.1394, 1) < 0.9974$. Since $h(x, 1)$ is a smooth function for $0 < x < 1$, there exists ν_1 such that $h(x, 1) < 0.9974$ for all $x \in [0.1394 - \nu_1, 0.1394]$. If $n > 1/\nu_1$, then there exists an $x_1 = x_1(n) \in [0.1394 - \nu_1, 0.1394]$ such that $x_1 n$ is an integer. By (21),

$$\frac{r(n,1,x_1 n)}{|\mathcal{F}_3(n)|} = O(n^3)(0.9974)^n \to 0 \qquad \text{as } n \to \infty.$$

By the definition of $r(n, k, x)$, (2) and Corollary 4, this implies (14).

Similarly, by plugging the corresponding values of x and k into (20), one can check that $0 < h(0.05, 2) < 0.9985$, $0 < h(0.0182, 3) < 0.9973$, $0 < h(0.0063, 4) < 0.9986$, and $0 < h(0.0022, 5) < 0.9979$. Thus repeating the argument of the previous paragraph, we obtain that (15), (16), (17), (18) also hold. ■

4. Bound on $|C_1 \cup C_2 \cup C_4|$

Definition 13. For a graph G, let $c_{1,2,4}(G)$ be the maximum size of $|C_1 \cup C_2 \cup C_4|$, where C_1, C_2 and C_4 are disjoint subsets of $V(G)$ such that C_i is i-independent for all $i \in \{1, 2, 4\}$.

In this section we prove an upper bound on $c_{1,2,4}(G)$ in terms of $c_1(G)$ for cubic graphs G of girth at least 9. For every vertex a in such a graph G, the ball $B_G(a, 2)$ induces a $(3, 2, a)$-tree T_a. When handling such a tree T_a, we will use the following notation (see Fig. 2):

$$V(T_a) = \{a\} \cup N_1(a) \cup N_2(a), \quad \text{where } N_1(a) = \{a_1, a_2, a_3\},\ N_2(a) = \{a_{1,1}, a_{1,2}, a_{2,1}, a_{2,2}, a_{3,1}, a_{3,2}\},$$

and

$$E(T) = \{aa_1, aa_2, aa_3, a_1a_{1,1}, a_1a_{1,2}, a_2a_{2,1}, a_2a_{2,2}, a_3a_{3,1}, a_3a_{3,2}\}.$$

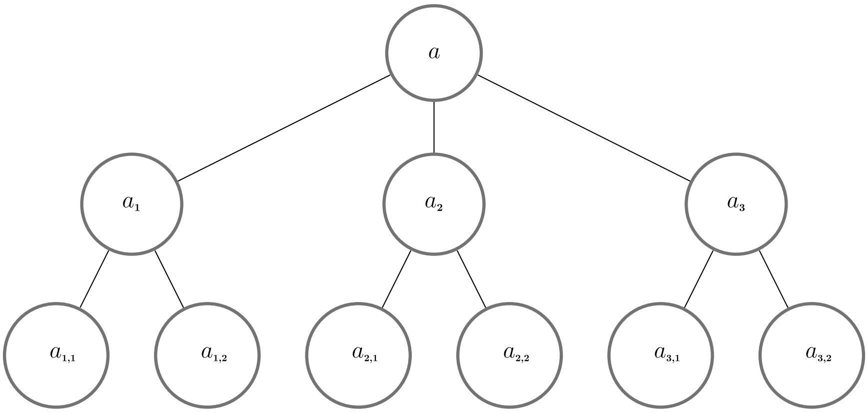

Fig. 2. A $(3, 2, a)$-tree T_a.

Lemma 14. *Let G be an n-vertex cubic graph with girth at least 9 and*

$$c_1(G) < 0.456n. \tag{22}$$

Then $c_{1,2,4}(G) \le 0.7174n =: b_{1,2,4}n$.

Proof. Let G satisfy the conditions of the lemma, and let C_1, C_2 and C_4 be disjoint subsets of $V(G)$ such that C_i is i-independent for $i \in \{1, 2, 4\}$ and $|C_1 \cup C_2 \cup C_4| = c_{1,2,4}(G)$.

The idea of the proof uses the fact that for a typical vertex $a \in C_4$, the tree T_a contains several vertices not in $C_1 \cup C_2$. For example, each vertex in G has at most one neighbor in C_2. Also for distinct $a_1, a_2 \in C_4$, the trees T_{a_1} and T_{a_2} are vertex-disjoint. For more accurate counting, we need a couple of new notions. Let Q be the set of vertices in C_1 that do not have neighbors in C_2, and $q = |Q|$. Let L be the set of edges in $G - C_1 - C_2$, and $\ell = |L|$. For brevity, the vertices in Q will be called *Q-vertices*, and the edges in L will be called *L-edges*. Let $s = |C_1| + |C_2|$. It will turn out that $q + \ell$ is a convenient parameter helping to bound $|C_4|$ in terms of s and $|C_2|$. We will prove the lemma in a series of claims. Our first claim is:

$$s < 0.652n. \tag{23}$$

To show (23), we count the edges connecting $C_1 \cup C_2$ with $\overline{C_1 \cup C_2}$ in two ways:

$$3(n - s) - 2\ell = e[C_1 \cup C_2, \overline{C_1 \cup C_2}] = 3s - 2(|C_1| - q). \tag{24}$$

Solving for s, we get $s = \frac{n}{2} - \frac{1}{3}(\ell - |C_1| + q)$. Since $q, \ell \ge 0$ and $|C_1| \le c_1$, this together with (22) yields

$$s \le \frac{n}{2} - \frac{1}{3}(0 - |C_1| + 0) \le \frac{n}{2} + \frac{c_1}{3} < 0.652n,$$

as claimed. ■

For $j \in \{0, 1, 2\}$, let

$$S_j = \{a \in C_4 : \text{ the total number of } L\text{-edges and } Q\text{-vertices in } T_a \text{ is } j\},$$

and let $U = C_4 - \bigcup_{j=0}^{2} S_j$.

Our next claim is:

$$\textit{For each } 0 \le j \le 2 \textit{ and every } a \in S_j,\ |V(T_a) \cap C_2| \ge 3 - j. \tag{25}$$

Indeed, let $0 \le j \le 2$ and $a \in S_j$. If a vertex $a_i \in N_1(a)$ is not in $(C_1 \cup C_2) - Q$, then either $a_i \in Q$ or $aa_i \in L$. Thus, by the definition of S_j, $|N_1(a) \cap ((C_1 \cup C_2) - Q)| \ge 3 - j$. Since each $a_i \in (C_1 \cup C_2) - Q$ either is in C_2 or has a neighbor in $C_2 \cap \{a_{i,1}, a_{i,2}\}$, we get at least $3 - j$ vertices in $C_2 \cap V(T_a)$. This proves (25).

For $0 \le j \le 2$, let $|S_j| = \alpha_j n$, and let $|U| = \beta n$. Then

$$(\alpha_1 + \alpha_2 + \alpha_3 + \beta)n = |C_4|. \tag{26}$$

By the definition of 4-independent sets, for all $a \in C_4$ the balls $B_G(a, 2)$ are disjoint and not adjacent to each other. For $0 \le j \le 2$ and every $a \in S_j$, the tree T_a contributes j to $\ell + q$, and for every $a \in U$, T_a contributes at least 3 to $\ell + q$. Therefore

$$\alpha_1 n + 2\alpha_2 n + 3\beta n \le \ell + q. \tag{27}$$

Also, (25) yields a lower bound on $|C_2|$:

$$3\alpha_0 n + 2\alpha_1 n + \alpha_2 n \le |C_2|. \tag{28}$$

Now (26), (27), and (28) yield

$$3|C_4| = (\alpha_1 n + 2\alpha_2 n + 3\beta n) + (3\alpha_0 n + 2\alpha_1 n + \alpha_2 n) \le \ell + q + |C_2|. \tag{29}$$

On the other hand, by (24)

$$2(\ell + q) = 3n - 6s + 2|C_1| = 3n - 4s - 2|C_2|,$$

so $2(\ell + q + |C_2|) = 3n - 4s$. Comparing with (29), we get

$$|C_4| \le \frac{3n - 4s}{6} = \frac{3n + 2s}{6} - s.$$

Hence by the definition of s and (23),

$$|C_1 \cup C_2 \cup C_4| = |C_4| + s \le \frac{3n + 2s}{6} \le \frac{n}{2} + \frac{0.652n}{3} \le 0.7174n. \quad \blacksquare$$

5. Proof of Theorem 1

For each fixed integer $k \ge 12$ and $g \ge 2k + 2$, let $J := \{3, 5, 6, 7, \ldots, 11\}$ and

$$\mathcal{B}_g(n) = \left\{ G \in \mathcal{G}_g(n) \ : \ c_{1,2,4}(G) + \sum_{j \in J} c_j(G) > 0.9785n \ \text{ or } \sum_{j=6}^{\lceil k/2 \rceil - 1} c_{2j+1}(G) > \frac{2 \cdot 0.45537n}{127} \right\}. \tag{30}$$

Lemma 15. *Let $k \ge 12$ be a fixed integer and $g \ge 2k + 2$. For every $\varepsilon > 0$, there exists an $N = N(\varepsilon) > 0$ such that for each $n > N$,*

$$|\mathcal{B}_g(n)| < \varepsilon \cdot |\mathcal{G}_g(n)|. \tag{31}$$

Proof. Let $\varepsilon > 0$ be given. By Lemma 14, Theorem 5, and Corollary 4, there exists an $N_{1,2,4} > 0$ such that for each $n > N_{1,2,4}$,

$$|\{G \in \mathcal{G}_g(n) \ : \ c_{1,2,4}(G) > b_{1,2,4} n\}| \ < \frac{\varepsilon}{10} \cdot |\mathcal{G}_g(n)|.$$

Let

$$M_{1,2,4}(n) := \{G \in \mathcal{G}_g(n) \ : \ c_{1,2,4}(G) > b_{1,2,4} n\}.$$

For each $j \in J$ and the constants b_j defined in Corollaries 10 and 12, let

$$M_j(n) := \{G \in \mathcal{G}_g(n) \ : \ c_j(G) > b_j n\}.$$

Let

$$\mathcal{B}'_g(n) = \left\{ G \in \mathcal{G}_g(n) : \ c_{1,2,4}(G) + \sum_{j \in J} c_j(G) > 0.9785n \right\}$$

$$\text{and} \quad \mathcal{B}''_g(n) = \left\{ G \in \mathcal{G}_g(n) : c_1(G) > 0.45537n \right\}.$$

If $G \in \mathcal{B}'_g(n)$, then

$$G \in M_{1,2,4}(n) \cup \bigcup_{j \in J} M_j(n),$$

because $b_{1,2,4} n + \sum_{j \in J} b_j n = 0.9785n$ and $c_{1,2,4} + \sum_{j \in J} c_j > 0.9785n$.

Corollaries 10 and 12 imply that for each $j \in J$, there exists an $N_j > 0$ such that for each $n > N_j$,

$$|\{G \in \mathcal{G}_g(n) \ : \ c_j(G) > b_j n\}| \ < \frac{\varepsilon}{10} \cdot |\mathcal{G}_g(n)|.$$

By Theorem 5, there exists an $N_1 > 0$ such that for each $n > N_1$, $|\mathcal{B}''_g(n)| \ < \frac{\varepsilon}{10} \cdot |\mathcal{G}_g(n)|$.

Let $N = \max\{N_{1,2,4}, N_1, N_3, N_5, N_6, \ldots, N_{11}\}$. By the definition of N, for each $n > N$,

$$|\mathcal{B}'_g(n)| + |\mathcal{B}''_g(n)| < (1 + |J| + 1)\frac{\varepsilon}{10} \cdot |\mathcal{G}_g(n)| = \varepsilon \cdot |\mathcal{G}_g(n)|. \tag{32}$$

Every graph $G \in \mathcal{G}_g(n) \setminus \mathcal{B}''_g(n)$ satisfies $c_1(G) \leq 0.45537n$. Using this, Lemma 8(ii) implies that such a graph G satisfies

$$\sum_{j=6}^{\lceil k/2 \rceil -1} c_{2j+1}(G) < \sum_{j=6}^{\lceil k/2 \rceil -1} \frac{c_1(G)}{2^{j+1}-1} < \sum_{j=6}^{\infty} \frac{0.45537n}{2^{j+1}-1} \leq \frac{0.45537n}{127} \cdot \sum_{s=0}^{\infty} \frac{1}{2^s} = \frac{2 \cdot 0.45537n}{127}.$$

It follows that $\mathcal{B}_g(n) \subseteq \mathcal{B}'_g(n) \cup \mathcal{B}''_g(n)$. Thus (32) implies (31). ■

Now we are prepared to prove our main result.

Proof of Theorem 1. Let $k \geq 12$ be a fixed integer and $g \geq 2k+2$. We need to show that for every $\varepsilon > 0$, there exists an $N > 0$ such that for each $n > N$,

$$|\{G \in \mathcal{G}_g(n) : \chi_p(G) \leq k\}| < \varepsilon \cdot |\mathcal{G}_g(n)|. \tag{33}$$

Let $\varepsilon > 0$ be given and $G \in \mathcal{G}_g(n)$ satisfy $\chi_p(G) \leq k$. Then there is a partition of $V(G)$ into $C_1, C_2, \ldots, C_k$ such that for each $i = 1, 2, \ldots, k$, C_i is i-independent. In particular, $|C_1| + |C_2| + \cdots + |C_k| = n$. By Lemma 8(i),

$$\sum_{j=6}^{\lfloor k/2 \rfloor} |C_{2j}| < \sum_{k=6}^{\infty} \frac{n}{3 \cdot 2^k - 2} < \frac{n}{190} \cdot \sum_{k=0}^{\infty} \frac{1}{2^k} = \frac{n}{95}. \tag{34}$$

Since $n - \frac{n}{95} > 0.9785n + \frac{2 \cdot 0.45537n}{127}$, this implies that $G \in \mathcal{B}_g(n)$, where $\mathcal{B}_g(n)$ is defined by (30). Thus, Lemma 15 implies (33). ■

6. Concluding remarks

1. It seems that with more sophisticated calculations, one can prove the claim of Theorem 1 not only for almost all cubic graphs with girth at least $2k+2$, but for almost all cubic n-vertex graphs. But we cannot prove (and maybe this is not true) that for each k *every* cubic graph G with sufficiently large (with respect to k) girth satisfies $\chi_p(G) > k$.
2. Our approach does not yield cubic *planar* graphs with high packing chromatic number. Gastineau, Holub and Togni [13] showed that $\chi_p(G)$ is bounded for graphs G in some subclasses of cubic *outerplanar* graphs.

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