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Strong edge-colorings of sparse graphs with large maximum degree

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ABSTRACT

A *strong k-edge-coloring* of a graph G is a mapping from $E(G)$ to $\{1, 2, \ldots, k\}$ such that every two adjacent edges or two edges adjacent to the same edge receive distinct colors. The *strong chromatic index* $\chi'_s(G)$ of a graph G is the smallest integer k such that G admits a strong k-edge-coloring. We give bounds on $\chi'_s(G)$ in terms of the maximum degree $\Delta(G)$ of a graph G when G is sparse, namely, when G is 2-degenerate or when the maximum average degree $\mathrm{Mad}(G)$ is small. We prove that the strong chromatic index of each 2-degenerate graph G is at most $5\Delta(G)+1$. Furthermore, we show that for a graph G, if $\mathrm{Mad}(G) < 8/3$ and $\Delta(G) \geq 9$, then $\chi'_s(G) \leq 3\Delta(G) - 3$ (the bound $3\Delta(G) - 3$ is sharp) and if $\mathrm{Mad}(G) < 3$ and $\Delta(G) \geq 7$, then $\chi'_s(G) \leq 3\Delta(G)$ (the restriction $\mathrm{Mad}(G) < 3$ is sharp).

1. Introduction

A *strong k-edge-coloring* of a graph G is a mapping from $E(G)$ to $\{1, 2, \ldots, k\}$ such that every two adjacent edges or two edges adjacent to the same edge receive distinct colors. In other words, the

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graph induced by each color class is an induced matching. The *strong chromatic index* of G, denoted by $\chi_s'(G)$, is the smallest integer k such that G admits a strong k-edge-coloring.

Strong edge-coloring was introduced by Fouquet and Jolivet [13,14] and was used to solve the frequency assignment problem in some radio networks. For more details on applications see [2,22–24].

An obvious upper bound on $\chi_s'(G)$ (given by a greedy coloring) is $2\Delta(G)(\Delta(G)-1)+1$ where $\Delta(G)$ denotes the maximum degree of G. Erdős and Nešetřil [10,11] conjectured that for every graph G with maximum degree Δ,

$$\chi_s'(G) \leq \begin{cases} \frac{5}{4}\Delta^2 & \text{if } \Delta \text{ is even} \\ \frac{5}{4}\Delta^2 - \frac{\Delta}{2} + \frac{1}{4} & \text{if } \Delta \text{ is odd.} \end{cases}$$

The bounds in the conjecture are sharp, if the conjecture is true.

The first nontrivial upper bound on $\chi_s'(G)$ was given by Molloy and Reed [21], who showed that $\chi_s'(G) \leq 1.998\Delta^2$, if Δ is sufficiently large. The coefficient 1.998 was improved to 1.93 (again, for sufficiently large Δ) by Bruhn and Joos [6]. Recently, Bonamy, Perrett and Postle [4] announced an even better coefficient of 1.835. For $\Delta = 3$, the conjecture was settled independently by Andersen [1] and by Horák, Qing and Trotter [16]. Cranston [9] proved that every graph with $\Delta \leq 4$ admits a strong edge-coloring with 22 colors, which is 2 more than the conjectured bound.

The strong chromatic index was studied for various families of graphs, such as cycles, trees, d-dimensional cubes, chordal graphs, and Kneser graphs, see [20]. There was also a series of papers [12,15,18] on strong edge-coloring planar graphs. In particular, Faudree, Gyárfás, Schelp and Tuza [12] proved that $\chi_s'(G) \leq 4\Delta+4$ for every planar graph G with maximum degree Δ and exhibited, for every integer $\Delta \geq 2$, a planar graph with maximum degree Δ and strong chromatic index $4\Delta-4$. Borodin and Ivanova [5] showed that every planar graph G with maximum degree $\Delta \geq 3$ and girth $g \geq 40\lfloor \Delta/2 \rfloor$ satisfies $\chi_s'(G) \leq 2\Delta-1$, and that the bound $2\Delta-1$ is sharp. Chang, Montassier, Pecher and Raspaud [7] relaxed the restriction on g to $g \geq 10\Delta+46$ for $\Delta \geq 4$.

Hudák, Lužar, Soták and Škrekovski [17] proved that $\chi_s'(G) \leq 3\Delta+6$ for every planar graph G with maximum degree $\Delta \geq 3$ and girth $g \geq 6$. Recently, Bensmail, Harutyunyan, Hocquard and Valicov [3] improved the upper bound $3\Delta+6$ for such graphs to $3\Delta+1$. With the stronger restriction of $g \geq 7$, Ruksasakchai and Wang [25] reduced the bound $3\Delta+1$ to 3Δ.

Clearly, planar graphs with large girth are sparse. The problem of strong edge-coloring was also studied for general sparse graphs. A natural measure of sparsity is *degeneracy*: a graph G is *d-degenerate* if every subgraph G' of G has a vertex of degree at most k (in G'). Chang and Narayanan [8] proved that $\chi_s'(G) \leq 10\Delta-10$ for every 2-degenerate graph G with maximum degree $\Delta \geq 2$. Luo and Yu [19] improved the bound $10\Delta-10$ to $8\Delta-4$. A more general bound by Yu [27] allowed to reduce the bound for 2-degenerate graphs to $6\Delta-5$, and Wang [26] improved it to $6\Delta-7$.

In this paper, we prove three bounds on the strong chromatic index of sparse graphs in terms of the maximum degree. Two of our bounds yield new bounds for planar graphs with girths 6 and 8.

Our first result is on 2-degenerate graphs. It improves the aforementioned bounds in [8,19,26] for $\Delta \geq 9$.

Theorem 1.1. *Every 2-degenerate graph G with maximum degree Δ satisfies $\chi_s'(G) \leq 5\Delta+1$.*

A finer measure of sparsity is the *maximum average degree*, denoted $\mathrm{Mad}(G)$, which is the maximum of $2\frac{|E(G')|}{|V(G')|}$ over all nontrivial subgraphs G' of a graph G. By definition, $\mathrm{Mad}(G) < 4$ for every 2-degenerate graph G. Two of our results show that if $\mathrm{Mad}(G) < 3$, then we can use significantly fewer than 5Δ colors. The graphs $K_\Delta(t)$ defined below show that our bounds are almost optimal. Let $K_\Delta(t)$ be the graph obtained from K_t by adding $\Delta-t+1$ pendant edges to each vertex in K_t. It is easy to check that $\mathrm{Mad}(K(t)) = t-1$ and $\chi_s'(K_\Delta(t)) = |E(K_\Delta(t))| = t\Delta - \binom{t}{2}$. In particular,

- $\mathrm{Mad}(K_\Delta(2)) = 1$ and $\chi_s'(K_\Delta(2)) = 2\Delta-1$,
- $\mathrm{Mad}(K_\Delta(3)) = 2$ and $\chi_s'(K_\Delta(3)) = 3\Delta-3$,
- $\mathrm{Mad}(K_\Delta(4)) = 3$ and $\chi_s'(K_\Delta(4)) = 4\Delta-6$.

Our second result is:

Theorem 1.2. *Let $\Delta \geq 9$ be an integer. Every graph G with maximum average degree less than $8/3$ and maximum degree at most Δ satisfies $\chi'_s(G) \leq 3\Delta - 3$.*

The graph $K_\Delta(3)$ above shows that the bound $3\Delta - 3$ is best possible. The graph $K'_\Delta(4)$ defined below shows that the bound on the maximum average degree is close to optimal:

We start from a copy R of K_4 with vertex set $\{v_1, v_2, v_3, v_4\}$ and let $K'_\Delta(4)$ be the graph obtained from R by subdividing the edge v_3v_4 with a vertex u and then adding $\Delta - 3$ pendant edges to each of v_1, v_2, v_3. It is not hard to check that the maximum degree of $K'_\Delta(4)$ is Δ, $\mathrm{Mad}(K'_\Delta(4)) = 14/5$, and $\chi'_s(K'_\Delta(4)) = 3\Delta - 2$.

Our last result is:

Theorem 1.3. *Let $\Delta \geq 7$ be an integer.*[1] *Every graph G with maximum average degree less than 3 and maximum degree at most Δ satisfies $\chi'_s(G) \leq 3\Delta$.*

Note that for small Δ, namely for $\Delta \leq 4$, the slightly weaker bound of $3\Delta + 1$ was proved by Ruksasakchai and Wang [25]. Since $\mathrm{Mad}(K_\Delta(4)) = 3$ and $\chi'_s(K_\Delta(4)) = 4\Delta - 6$, the restriction on the maximum average degree in Theorem 1.3 is best possible for $\Delta \geq 7$. The graph $K'_\Delta(4)$ above with $\mathrm{Mad}(K'_\Delta(4)) = 14/5$ and $\chi'_s(K'_\Delta(4)) = 3\Delta - 2$ shows that the bound 3Δ is also close to the best possible.

Since $\mathrm{Mad}(G) < \frac{2g}{g-2}$ for every planar graph G with girth g, Theorem 1.2 yields that $\chi'_s(G) \leq 3\Delta - 3$ for every planar graph G with maximum degree $\Delta \geq 9$ and girth $g \geq 8$ and Theorem 1.3 implies that $\chi'_s(G) \leq 3\Delta$ for every planar graph G with maximum degree $\Delta \geq 7$ and girth $g \geq 6$. The last result improves the bounds in [3,17,25] mentioned above for $\Delta \geq 7$.

The structure of the paper is as follows. In Section 2 we introduce some notation and prove useful lemmas. In Sections 3–5, we prove Theorems 1.1–1.3, respectively.

2. Notation and preliminaries

Let $[k] := \{1, \ldots, k\}$. For a function f defined on a set A' with $A' \subseteq A$, we denote $f(A) := f(A') = \{f(a) : a \in A'\}$. For a graph G, let $\overline{d}(G)$ be the average degree of G. We define $\mathrm{Mad}(G) := \max_{H \subseteq G} \overline{d}(H)$. A vertex $v \in V(G)$ is a *d^+_G-vertex* if $d_G(v) \geq d$. If G is clear from the context, then we simply say that v is a *d^+-vertex*. A *d^+_G-neighbor* of a vertex $v \in V(G)$ is a neighbor of v that is a d^+_G-vertex. A *d^-_G-vertex*, a *d_G-vertex*, a *d^-_G-neighbor*, and a *d_G-neighbor* are defined similarly.

For a vertex $v \in V(G)$, let $N_G(v)$ denote the set of all neighbors of v in G and let $\Gamma_G(v)$ denote the set of all edges incident to v in G. For an edge $e = uv$, let

$$N_G[e] := \Gamma_G(u) \cup \Gamma_G(v) \text{ and } N^2_G[e] := \bigcup_{w \in N_G(u) \cup N_G(v)} \Gamma_G(w).$$

A function $f : E(G) \to [k]$ is a *strong k-edge-coloring* of G if $f(e) \neq f(e')$ for any $e, e' \in E(G)$ with $e' \in N^2_G[e] \setminus \{e\}$. Since below we only consider strong edge-colorings, for brevity we will simply call them *colorings*. A function $f : E' \to [k]$ is a *partial k-coloring* of G on E' if $E' \subseteq E(G)$ and $f(e) \neq f(e')$ for any $e, e' \in E'$ with $e' \in N^2_G[e] \setminus \{e\}$. For a partial k-coloring $f : E' \to [k]$ of a graph G and $e \in E(G)$, let the *f-multiplicity of e*, $m(f, e)$, be $m(f, e) := |N^2_G[e] \cap E'| - |f(N^2_G[e])|$. Note that $m(f, e)$ counts multiple occurrences of all colors in $N^2_G[e]$. By definition, $|f(N^2_G[e])| = |N^2_G[e] \cap E'| - m(f, e)$. In particular, if $N^2_G[e]$ contains two edges with the same color, then $m(f, e) \geq 1$.

For a partial k-coloring $f : E \to [k]$ and $e', e'' \in E$, we often say "we extend f to e' by coloring it with a color α". This means that we replace f with a new function $f' : E \cup \{e'\} \to [k]$ such that $f'(e) := f(e)$ for all $e \in E \setminus \{e'\}$ and $f'(e') := \alpha$. Also, we say "we switch the colors of e and e'" when we replace f with a new function $f' : E \to [k]$ such that $f'(e) := f(e)$ for all $e \in E \setminus \{e', e''\}$, $f'(e') := f(e'')$ and $f'(e'') := f(e')$. In both cases, we will slightly abuse the notation by denoting the new updated function by f.

[1] We can prove the result for $\Delta \geq 6$, but that would make the proof longer and more complicated.

For a partial k-coloring f of G, we say that a sequence $(E_1, E_2, \dots, E_s)$ of pairwise disjoint subsets of $E(G)$ is an *(f, k)-degenerate sequence for G* if the following holds:

- $f : E(G) \setminus (\bigcup_{i=1}^{s} E_i) \to [k]$ is a partial k-coloring of G.
- For every $i \in [s]$ and $e \in E_i$, $|N_G^2[e] \setminus \bigcup_{j=i+1}^{s} E_j| \le k + m(f, e)$.

Note that if $(E_1, E_2, \dots, E_s)$ is an (f, k)-degenerate sequence for G, then the domain of f is exactly $E(G) \setminus (\bigcup_{i=1}^{s} E_i)$, thus $\bigcup_{i=1}^{s} E_i$ is exactly the set of all edges of G uncolored by f. For a partial k-coloring f of a graph G, the graph G is *(f, k)-degenerate* if there exists an (f, k)-degenerate sequence $(E_1, \dots, E_s)$ for G. If $E_i = \{e_i\}$, then for simplicity, instead of $(E_1, \dots, E_s)$, we write $(E_1, \dots, E_{i-1}, e_i, E_{i+1}, \dots, E_s)$.

The following lemma regarding degeneracy is useful.

Lemma 2.1. *If a graph G has a partial k-coloring f and is (f, k)-degenerate, then $\chi_s'(G) \le k$.*

Proof. Assume we have a partial k-coloring f of G with domain E_0 and $(E_1, \dots, E_s)$ is an (f, k)-degenerate sequence on G. Let $S = (e_1, \dots, e_t)$ be an ordering of all edges in $\bigcup_{i=1}^{s} E_i$ such that for $j \in [s-1]$, all edges in E_j come before any edge in E_{j+1}.

We iteratively color edges in S in order to extend f to $f_1, \dots, f_t$. Assume that we have colored $e_1, \dots, e_{i-1}$ for $i \in [t]$ and have a partial k-coloring f_{i-1} on $E_0 \cup \{e_1, \dots, e_{i-1}\}$. Let $e_i \in E_j$. Note that $m(f, e) \le m(f_{i-1}, e)$ for any $e \in E(G)$. Then since $(E_1, \dots, E_s)$ is an (f, k)-degenerate sequence for G,

$$\left|f_{i-1}(N_G^2[e_i])\right| = \left|N_G^2[e_i] \setminus \{e_i, \dots, e_t\}\right| - m(f_{i-1}, e_i) \le \left|N_G^2[e_i] \setminus \left(\{e_i\} \cup \bigcup_{\ell=j+1}^{s} E_\ell\right)\right| - m(f, e_i)$$

$$= \left|N_G^2[e_i] \setminus \bigcup_{\ell=j+1}^{s} E_\ell\right| - |\{e_i\}| - m(f, e_i) \le k - 1.$$

Thus we can choose a color $c \in [k] \setminus f_{i-1}(N_G^2[e_i])$. Let

$$f_i(e) := \begin{cases} f_{i-1}(e) & \text{if } e \in E_0 \cup \{e_1, \dots, e_{i-1}\}, \\ c & \text{if } e = e_i. \end{cases}$$

Now, f_i is a partial k-coloring of G with domain $E_0 \cup \{e_1, \dots, e_i\}$. By repeating this process, we get a strong k-edge-coloring of G. Thus $\chi_s'(G) \le k$. □

We say a vertex u is *pale in G* if u has at most two 3_G^+-neighbors. A vertex u is *light in G* if all vertices in $N_G(u)$ except at most two are pale. Since a vertex with degree two is pale, each pale vertex is also light.

Lemma 2.2. *If G' is a subgraph of a graph G with $\delta(G') \ge 3$, then G' has no vertex that is light in G.*

Proof. Assume G contains a subgraph G' with $\delta(G') \ge 3$ and $v \in V(G')$ is light in G. Then $|N_{G'}(v)| \ge 3$. So by the definition of "light", there is a pale $w \in N_{G'}(v)$. Similarly, $|N_{G'}(w)| \ge 3$. So by the definition of "pale", some $u \in N_{G'}(w)$ is a 2_G^--vertex, contradicting $\delta(G') \ge 3$. □

Lemma 2.3. *Every 2-degenerate graph G with $\Delta(G) \ge 3$ has a vertex v of degree at least three that is adjacent to at most two vertices of degree at least three.*

Proof. Let V' denote the set of 3^+-vertices. Since $\Delta(G) \ge 3$, we know that $V' \ne \emptyset$. Since G is 2-degenerate, $G[V']$ is also 2-degenerate. In particular, $G[V']$ has a vertex v of degree at most 2. This v satisfies the lemma. □

3. 2-degenerate graphs

In this section we prove the following stronger result, which implies Theorem 1.1.

Theorem 3.1. *Let a 2-degenerate graph G and a positive integer $D \geq 2$ satisfy the following:*

(A1) $\Delta(G) \leq D + 2$.
(A2) *For $t \in [2]$, every vertex $v \in V(G)$ with $d_G(v) = D + t$ is adjacent to at least t vertices of degree one.*

Then $\chi'_s(G) \leq 5D + 1$.

Proof. Let G be a counterexample to the statement with the fewest 2^+-vertices, and subject to this, with the fewest edges. Let G^* be obtained from G by deleting all vertices of degree 1 in G. By (A1) and (A2), $\Delta(G^*) \leq D$.

First we prove that

$$\text{if } v \in V(G^*) \text{ and } d_{G^*}(v) \leq 2, \text{ then } d_{G^*}(v) = d_G(v). \tag{3.1}$$

Indeed, suppose to the contrary that $N_{G^*}(v) = \{w_1, \dots, w_t\}$ where $t \in [2]$ and there is $u \in N_G(v) \setminus N_{G^*}(v)$ with $d_G(u) = 1$. The graph $G' := G - u$ is 2-degenerate and has no more $2^+_{G'}$-vertices than G has 2^+_G-vertices. G' also satisfies (A1) and (A2). Furthermore, G' has strictly fewer edges than G. By the minimality of G, the graph G' has a $(5D + 1)$-coloring f. Since

$$|N^2_G[uv]| \leq \left| \Gamma_G(v) \cup \bigcup_{i=1}^{t} \Gamma_G(w_i) \right| \leq D + 2 + D + 2 + D + 2 - 2 \leq 5D + 1,$$

when $D \geq 2$, we can extend f to uv, a contradiction. This proves (3.1). In particular, (3.1) yields

$$\delta(G^*) \geq 2. \tag{3.2}$$

If $\Delta(G^*) \leq 2$, then by (3.2), G^* is a disjoint union of cycles. Then by (3.1), G itself is a disjoint union of cycles. So by the minimality of G, it is a cycle. Since each edge of a cycle has at most four edges at distance at most 2, it is strong 5-edge-colorable. This contradicts the choice of G since $5 \leq 5D + 1$. Thus $\Delta(G^*) > 2$. Then by Lemma 2.3, G^* has a vertex v with $d_{G^*}(v) \geq 3$ that is adjacent to at most two $3^+_{G^*}$-vertices. We fix such a vertex to be v and let $N_{G^*}(v) = \{v_1, v_2, u_1, \dots, u_t\}$ where $d_{G^*}(u_i) = 2$ for all $i \in [t]$. (It could be that $d_{G^*}(v_1) = 2$ and/or $d_{G^*}(v_2) = 2$.)

By the choice of v, we know $t \geq 1$. By (3.1) and (3.2), $d_{G^*}(u_i) = d_G(u_i) = 2$ for each $i \in [t]$. For each $i \in [t]$, let $N_G(u_i) = \{v, w_i\}$. By (A1) and (A2), we have

$$t + 2 \leq D. \tag{3.3}$$

We claim that

$$d_{G^*}(v) = d_G(v). \tag{3.4}$$

Indeed, suppose to the contrary that $u \in N_G(v) \setminus N_{G^*}(v)$ with $d_G(u) = 1$. The graph $G'' := G - u$ is 2-degenerate and has no more $2^+_{G''}$-vertices than G has 2^+_G-vertices. G'' also satisfies (A1) and (A2). Furthermore, G'' has strictly fewer edges than G. By the minimality of G, the graph G'' has a $(5D + 1)$-coloring f. Since

$$\begin{aligned}|N^2_G[uv]| &\leq d_G(v) + \sum_{i=1}^{2}(d_G(v_i) - 1) + \sum_{i=1}^{t}(d_G(u_i) - 1) \overset{(3.1)}{\leq} (D + 2) + 2(D + 1) \\ &\quad + t \overset{(3.3)}{\leq} 5D + 1,\end{aligned}$$

when $D \geq 1$, we can extend f to uv, a contradiction. This proves (3.4).

Suppose w_1 has exactly h neighbors of degree 1 in G. Let H be the graph obtained from $G - vu_1$ by adding $\ell := \max\{0, 2-h\}$ new vertices $x_1, \dots, x_\ell$, each of which is adjacent only to w_1. Since the degree of u_1 in G is 2 and in H is 1, H has fewer 2^+-vertices than G. Also by construction,

$$w_1 \text{ has at least 3 neighbors of degree 1 in } H\text{, say } u_1, u_1', \text{ and } u_1''. \tag{3.5}$$

It is not hard to check that H inherits properties (A1) and (A2) from G. Note that H has fewer 2^+-vertices than G. So, by the minimality of G, the 2-degenerate graph H has a $(5D+1)$-coloring f. By (3.5), we can switch the colors of w_1u_1, w_1u_1', and w_1u_1'' so that

$$f(w_1u_1) \notin \{f(vv_1), f(vv_2)\}. \tag{3.6}$$

Case 1. $f(w_1u_1) \notin \{f(vu_2), \dots, f(vu_t)\}$. Together with (3.4) and (3.6), this yields that $f|_{E(G)}$ is a partial coloring of G where only vu_1 is not colored. Since

$$\begin{aligned} |N_G^2[vu_1]| &\le d_G(v_1) + d_G(v_2) + (d_G(w_1) - 1) + \sum_{i=1}^{t} d_G(u_i) \\ &\le (D+2) + (D+2) + (D+1) + 2t \overset{(3.3)}{\le} 3D + 5 + 2D - 4 \le 5D + 1, \end{aligned} \tag{3.7}$$

we can extend f to vu_1.

Case 2. There is $i \in [t] \setminus \{1\}$ such that $f(w_1u_1) = f(vu_i)$. Then by (3.4) and (3.7), we can choose $f(vu_1)$ so that the only conflict in f will be that $f(w_1u_1) = f(vu_i)$. Let f' be obtained from f by uncoloring vu_i. Then we have Case 1 with f' in place of f and u_i in place of u_1. This proves the theorem. □

4. Graphs with maximum average degree less than 8/3

In this section we prove Theorem 1.2:
If $\Delta \ge 9$ and G is a graph with $\mathrm{Mad}(G) < 8/3$ *and $\Delta(G) \le \Delta$, then $\chi_s'(G) \le 3\Delta(G) - 3$.*

Similarly to the proof of Theorem 3.1, consider a counterexample G with the fewest 2^+-vertices, and subject to this with the fewest edges. Let G^* be the graph obtained from G by deleting all vertices of degree 1. Let $\Delta = \Delta(G)$.

Claim 4.1. $\delta(G^*) \ge 2$.

Proof. Suppose u is a 1_{G^*}-vertex where w is the neighbor of u in G^*. Then there exists $v \in N_G(u) - w$ with $d_G(v) = 1$. By the minimality of G, the graph $G - v$ has a $(3\Delta - 3)$-coloring f. Then f is a partial $(3\Delta - 3)$-coloring of G, and since

$$|N_G^2[uv]| = \sum_{x \in N_G(u)} d_G(x) \le d_G(w) + |N_G(u) \setminus N_{G*}(u)| \le \Delta + (\Delta - 1) \le 3\Delta - 3, \tag{4.1}$$

we can extend f to uv, a contradiction. □

Say that a vertex v is *special* if $d_{G^*}(v) = 2$ and v is adjacent to a $(\Delta - 1)^+_{G^*}$-vertex.

Claim 4.2. *If u_1 and u_2 are two adjacent 2_{G^*}-vertices, then both u_1 and u_2 are special.*

Proof. Let $N_{G^*}(u_1) = \{w_1, u_2\}$ and $N_{G^*}(u_2) = \{u_1, w_2\}$. Assume u_1 is not special, so that $d_{G^*}(w_1) \le \Delta - 2$.

Obtain the graph H from G by first deleting all 1_G-neighbors of u_1 and u_2, then deleting the edge u_1u_2, and then adding leaves adjacent to w_1 and w_2 so that $d_H(w_i) = \Delta$ for $i \in [2]$. By construction, H has fewer vertices of degree at least 2 than G. Also, either $\mathrm{Mad}(G) < 2$ and hence $\mathrm{Mad}(H) < 2$ or $\mathrm{Mad}(H) \le \mathrm{Mad}(G)$ and hence $\mathrm{Mad}(H) < 8/3$. So H has a $(3\Delta - 3)$-coloring f by the minimality of G. Also w_1 is incident to at least three pendant edges in H, one of which is u_1w_1. Let w_1v_1 and w_1v_2 be two other such edges.

By the definition of H, $|f(\Gamma_H(w_1))| = |f(\Gamma_H(w_2))| = \Delta$. We define a special color α as follows. If $f(w_2u_2) \in f(\Gamma_H(w_1))$, then let $\alpha := f(w_2u_2)$. Otherwise, $f(\Gamma_H(w_1)) \setminus f(\Gamma_H(w_2)) \neq \emptyset$, and we let α be any color in this difference. After this, we switch the colors of u_1w_1 and w_1v_1, if necessary, so that $f(u_1w_1) \neq \alpha$. In particular, this means $f(u_1w_1) \neq f(u_2w_2)$, $f|_{E(G)}$ is a partial coloring of G.

We extend f to u_1u_2 by coloring it with a color not in $f(\Gamma_H(w_1) \cup \Gamma_H(w_2))$, which is possible since $2\Delta + 1 \leq 3\Delta - 3$. Now we will color the pendant edges of G incident with u_2, if there are any. In particular, if there is at least one such edge u_2u_2' and $\alpha \neq f(w_2u_2)$, then we start by letting $f(u_2u_2') = \alpha$. After coloring all pendant edges incident to u_2, we have

$$d_G(u_2) = 2 \text{ or } N_G^2[u_1z] \text{ has two edges of color } \alpha \text{ for each } 1_G\text{-neighbor } z \text{ of } u_1. \tag{4.2}$$

Then we color the remaining edges in $\Gamma_G(u_2)$ one by one, since the only colored edges that could be in conflict are in $\Gamma_G(w_2) \cup \Gamma_G(u_2) \cup \{w_1u_1\}$, and there are at most $2\Delta + 1$ such edges. Finally, we remove the edges in $E(H) - E(G)$ and color the pendant edges of G incident with u_1 one by one. For every pendant edge $u_1z \in E(G)$ incident with u_1, at the moment of coloring u_1z, the number $M(u_1z)$ of the colors forbidden for u_1z is at most $|N_G^2[u_1z] \setminus \{u_1z\}|$. Moreover, by (4.2), if $d_G(u_2) \geq 3$, then $M(u_1z) \leq |N_G^2[u_1z] \setminus \{u_1z\}| - 1$. In any case,

$$\begin{aligned} M(u_1z) &\leq d_G(w_1) + \max\{2, d_G(u_2) - 1\} + (d_G(u_1) - 3) \leq \Delta + \max\{2, \Delta - 1\} + (\Delta - 3) \\ &= 3\Delta - 4. \end{aligned}$$

So we can always find a free color for u_1z. This yields a $(3\Delta - 3)$-coloring of G, a contradiction. This shows that u_1 is special, and by symmetry, this also shows that u_2 is special. $\square$

Claim 4.3. *If u is a 3_{G^*}-vertex with two 2_{G^*}-neighbors v_1 and v_2, then at least one of v_1, v_2 is special.*

Proof. Let $N_{G^*}(v_i) = \{u, v_i'\}$ for $i \in [2]$ and let $N_{G^*}(u) = \{w, v_1, v_2\}$. Suppose to the contrary that both v_1, v_2 are not special, so v_1', v_2' are both $(\Delta-2)^-_{G^*}$-vertices. We construct a graph H from $G \setminus \{uv_1, uv_2\}$ by deleting all 1_G-neighbors of u, v_1, and v_2 and then adding leaves to v_1', v_2', and w to make the degrees of v_1', v_2', and w equal to Δ. Since $\mathrm{Mad}(G) < 8/3$, and we were adding only 1-vertices, $\mathrm{Mad}(H) < 8/3$. Since u, v_1, and v_2 are leaves in H but not in G, the graph H has fewer 2^+-vertices than G. Thus by the minimality of G, the graph H has a $(3\Delta - 3)$-coloring f.

For $i \in [2]$, since v_i' is a $(\Delta-2)^-_{G^*}$-vertex, it is adjacent to at least three leaves in H, including v_i; let $v_{i,1}'$ and $v_{i,2}'$ be two other leaves adjacent to v_i' in H.

We now define special colors α_1 and α_2 as follows. For $i \in [2]$, if $f(uw) \in f(\Gamma_H(v_i'))$, then we let $\alpha_i := f(uw)$. Otherwise, the set $f(\Gamma_H(v_i')) \setminus f(\Gamma_H(w))$ is nonempty, and we let α_i be any color in this difference. Note that $\alpha_2 = \alpha_1$ is possible. By definition,

$$|\{\alpha_1, \alpha_2, f(uw)\} \cap f(\Gamma_H(v_i'))| \leq 2 \quad \text{for } i \in [2]. \tag{4.3}$$

Also, for $i \in [2]$, the colors $f(v_i'v_i), f(v_i'v_{i,1}'), f(v_i'v_{i,2}')$ are all distinct. So by (4.3), at least one of them is not in $\{\alpha_1, \alpha_2, f(uw)\}$. Hence we can switch these colors so that $f(v_i'v_i) \notin \{\alpha_1, \alpha_2, f(uw)\}$. Then $f|_{E(G)}$ is a partial $(3\Delta - 3)$-coloring of G.

Let $H' = H + v_1u + v_2u$. Then

$$|N_{H'}^2[uv_i]| \leq |\Gamma_{H'}(w)| + |\Gamma_{H'}(v_i')| + |\Gamma_{H'}(v_{3-i})| + 1 \leq \Delta + \Delta + 2 + 1 \leq 3\Delta - 3.$$

Thus for $i \in [2]$, we can extend f to uv_i by coloring it with a color not in $f(N_{H'}^2[uv_i])$.

Now we will color the pendant edges of G incident with u, if there are any. At first, if there are at least two such edges uu_1', uu_2' then we do the following : if $\alpha_1 \neq f(wu)$, then we let $f(uu_1') = \alpha_1$, if $\alpha_2 \notin \{f(uw), \alpha_1\}$, then we let $f(uu_2') = \alpha_2$. Then we color the remaining pendant edges of G incident with u, if there are any. After coloring all pendant edges incident to u, for $i \in [2]$,

$$d_G(u) \leq 4 \text{ or } N_G^2[v_iz] \text{ contains two edges of color } \alpha_i \text{ for every } 1_G\text{-neighbor } z \text{ of } v_i. \tag{4.4}$$

Then we color the remaining edges in $\Gamma_G(u)$ one by one, since the only colored edges that could be in conflict are in $\Gamma_G(w) \cup \Gamma_G(u) \cup \{v_1v_1', v_2v_2'\}$, and there are at most $2\Delta + 1$ such vertices. Finally, we remove the edges in $E(H) \setminus E(G)$ and for $i \in [2]$ color the leaves of G incident with v_i one by one. For every leaf $v_iz \in E(G)$ incident with v_i, at the moment of coloring v_iz, the number $M(v_iz)$ of the colors forbidden for v_iz is at most $|N_G^2[v_iz] \setminus \{v_iz\}|$. Moreover, by (4.4), if $d_G(u) \geq 5$, then $M(v_iz) \leq |N_G^2[v_iz] \setminus \{v_iz\}| - 1$. In any case,

$$\begin{aligned} M(v_iz) &\leq d_G(v_i') + \max\{4, d_G(u) - 1\} + (d_G(v_i) - 3) \leq \Delta + \max\{4, \Delta - 1\} + (\Delta - 3) \\ &= 3\Delta - 4. \end{aligned}$$

So we can always find a free color for v_iz, for each $i \in [2]$. This yields a $(3\Delta - 3)$-coloring of G, a contradiction. □

Now we will complete the proof of the theorem using discharging: For each $v \in V(G^*)$, we let the *initial charge* $\mathrm{ch}(v) := d_{G^*}(v)$, and then will move charge among vertices so that the *final charge* $\mathrm{ch}^*(v)$ is at least $8/3$ for each $v \in V(G^*)$, but the total sum of charge will be preserved during the entire process. This will imply that

$$\sum_{v \in V(G^*)} d_{G^*}(v) = \sum_{v \in V(G^*)} ch(v) = \sum_{v \in V(G^*)} ch^*(v) \geq \frac{8}{3}|V(G^*)|, \tag{4.5}$$

contradicting the fact that $\mathrm{Mad}(G^*) < \frac{8}{3}$.

The rules of discharging are the following.

(R1) Each $(\Delta - 1)^+_{G^*}$-vertex sends charge $2/3$ to each neighbor.
(R2) Each vertex v with $4 \leq d_{G^*}(v) \leq \Delta - 2$ sends charge $1/3$ to every 2_{G^*}-neighbor.
(R3) Each 3_{G^*}-vertex sends charge $1/3$ to every 2_{G^*}-neighbor that is not special.

By (4.5), the theorem will follow from the following claim:

Claim 4.4. *For every $v \in V(G^*)$, $ch^*(v) \geq \frac{8}{3}$.*

Proof. We consider several cases depending on the degree of v.

If v is a special 2_{G^*}-vertex, then it receives charge $2/3$ from its $(\Delta - 1)^+_{G^*}$-neighbor by Rule (R1) and gives out nothing. Thus $\mathrm{ch}^*(v) \geq \mathrm{ch}(v) + 2/3 = 8/3$.

If v is a 2_{G^*}-vertex but not special, then by Claim 4.2 it has two $3^+_{G^*}$-neighbors, and each of them sends v charge $1/3$ either by (R2) or by (R3). Thus $\mathrm{ch}^*(v) \geq \mathrm{ch}(v) + 1/3 + 1/3 = 8/3$.

If $d_{G^*}(v) = 3$ and v is adjacent to exactly t vertices of degree two, then by Claim 4.3, at least $t - 1$ of them are special. Thus by Rule (R3), v sends out charge $1/3$ to at most one of its neighbors. Hence $\mathrm{ch}^*(v) \geq \mathrm{ch}(v) - 1/3 = 8/3$.

If $4 \leq d_{G^*}(v) \leq \Delta - 2$, then by (R2), v sends charge at most $1/3$ to each of its neighbors. Thus $\mathrm{ch}^*(v) \geq \mathrm{ch}(v) - \frac{d_{G^*}(v)}{3} \geq 2\mathrm{ch}(v)/3 \geq 8/3$.

Finally, if $d_{G^*}(v) \geq \Delta - 1$, then by (R1), v sends charge $2/3$ to each of its neighbors. Thus $\mathrm{ch}^*(v) \geq \mathrm{ch}(v) - \frac{2d_{G^*}(v)}{3} = \mathrm{ch}(v)/3 \geq (\Delta - 1)/3 \geq 8/3$ since $\Delta \geq 9$. □

5. Graphs with maximum average degree less than three

In this section, instead of Theorem 1.3 we prove the following stronger result.

Theorem 5.1. *Let $\Delta \geq 7$ be an integer and let G be a graph with no 3-regular subgraph. If $\Delta(G) \leq \Delta$ and $\mathrm{Mad}(G) \leq 3$, then $\chi_s'(G) \leq 3\Delta$.*

5.1. Set-up of the proof and some notation

To prove Theorem 5.1, we consider a counterexample G with the fewest 2^+-vertices, and subject to this with the fewest edges. Let G^* be the graph obtained from G by deleting all vertices of degree 1. Similarly to the proof of Theorem 1.2, we will show that vertices of "low" degree in G^* have neighbors

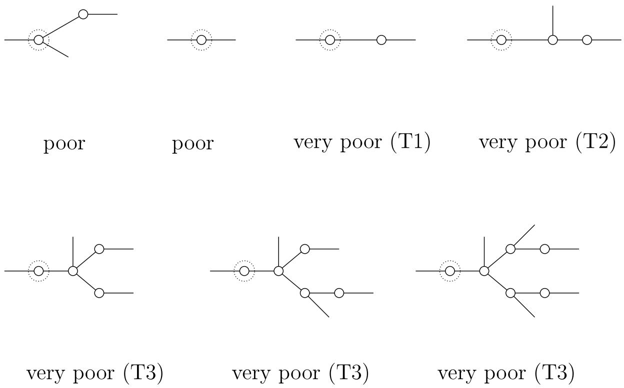

Fig. 1. Poor vertices and very poor vertices.

with "high" degree, and based on this use discharging to prove that the average degree of G^* is greater than 3.

A feature not used in the previous proofs is the notion of potentials. For a graph G and $A \subseteq V(G)$, the *potential* of A, denoted $\rho_G(A)$, is defined as

$$\rho_G(A) := 3|A| - 2|E(G[A])|.$$

By definition, $\text{Mad}(G) \leq 3$ if and only if $\rho_G(A) \geq 0$ for all $A \subseteq V(G)$.

The following fact on potentials is easy to check.

Lemma 5.2. *For a graph G and any disjoint $A, B \subset V(G)$, $\rho_G(A) + \rho_G(B) = \rho_G(A \cup B) + \rho_G(A \cap B) + 2|E_G(A \setminus B, B \setminus A)|$.*

A 3_{G^*}-vertex is *poor* if it has exactly one 2_{G^*}-neighbor. For a poor 3_{G^*}-vertex u, an edge $uw \in E(G^*)$ is the *u-sink* if $d_{G^*}(w) = 2$ and $N_{G^*}(u) - w$ contains a vertex u' with $d_G(u') < \Delta$. An edge is a *sink* if it is a u-sink for some poor 3_{G^*}-vertex u. By definition, a poor 3_{G^*}-vertex that is adjacent to two Δ_{G^*}-vertices is not incident to a sink.

Let u be a 2_{G^*}-vertex with $N_{G^*}(u) = \{v, w\}$. We say that u is very poor, v is a *sponsor of* u, and w is a *rival of* u, if one of the following holds:

(T1) w is a 2_{G^*}-vertex, or
(T2) w is a 3_{G^*}-vertex with two 2_{G^*}-neighbors in G^* (including u), or
(T3) w is a 4_{G^*}-vertex and each vertex in $N_{G^*}(w)$ except at most one is either a 2_{G^*}-vertex or a poor 3_{G^*}-vertex.

If a 2_{G^*}-vertex is not very poor, then we say that it is *poor*. See Fig. 1 for an illustration. Poor vertices will be the recipients of charge in the discharging procedure.

If u is a very poor vertex with a sponsor v and a rival w, then the edge uw is a *lower link* of u. Furthermore, if w is the rival of type (T3) of u and w' is a poor 3_{G^*}-vertex in $N_{G^*}(w)$, then ww' a *semi-link* of w.

Let $B(G)$ denote the set of all sinks in G. Similarly, let $S(G)$ and $S'(G)$ denote the set of all lower links in G and the set of all semi-links in G, respectively. By definition, all poor vertices and all very poor vertices are light. Also the rival of each very poor vertex is light.

5.2. Structure of G and G^*

The next claim is in the spirit of Claims 4.2 and 4.3.

Claim 5.3. *Let $v \in V(G^*)$.*

(i) *If*

$$\sum_{u \in N_{G^*}(v)} d_G(u) \leq 2\Delta + d_{G^*}(v), \tag{5.1}$$

then $d_G(v) = d_{G^*}(v)$.

(ii) *$d_{G^*}(v) \geq 2$.*

(iii) *If $d_{G^*}(v) = 2$, then $d_G(v) = 2$.*

(iv) *If $d_{G^*}(v) = 3$ and v has a neighbor $u \in V(G^*)$ with $d_G(u) \leq 3$, then $d_G(v) = 3$. In particular, if v is poor 3_{G^*}-vertex, then $d_G(v) = 3$.*

By (ii), *$\delta(G^*) \geq 2$.*

Proof. (i) Suppose (5.1) holds. If $d_G(v) \neq d_{G^*}(v)$, then v has a 1_G-neighbor w. By the minimality of G, the graph $G - w$ has a 3Δ-coloring f, which is a partial 3Δ-coloring of G. However by (5.1),

$$|N_G^2[vw]| \leq d_G(v) + \sum_{u \in N_G(v)} (d_G(u) - 1) = d_G(v) + \sum_{u \in N_{G^*}(v)} d_G(u) - d_{G^*}(v) \leq d_G(v) + 2\Delta \leq 3\Delta.$$

Thus we can extend f to vw, a contradiction to the definition of G. This proves (i).

(ii) Assume that v is a 1_{G^*}-vertex. Let u be the unique neighbor of v in G^*. Since $v \in V(G^*)$, we have $d_G(v) \geq 2 > d_{G^*}(v)$. However, $\sum_{w \in N_{G^*}(v)} d_G(w) = d_G(u) \leq \Delta$, and (5.1) holds. Together with $d_G(u) > d_{G^*}(u)$, this contradicts (i). This proves (ii), which implies $d_{G^*}(v) \geq 2$.

(iii) If $d_{G^*}(v) = 2$, then $\sum_{u \in N_{G^*}(v)} d_G(u) \leq 2\Delta$. Thus (i) implies (iii).

(iv) If $d_{G^*}(v) = 3$ and v has a neighbor $u \in V(G^*)$ with $d_G(u) \leq 3$, then

$$\sum_{w \in N_{G^*}(v)} d_G(w) \leq (d_{G^*}(v) - 1)\Delta + d_G(u) \leq 2\Delta + 3.$$

Thus (5.1) holds, and (i) implies that $d_G(v) = d_{G^*}(v) = 3$. The "In particular" part follows from (ii) and the definition of a poor 3_{G^*}-vertex. $\square$

The next claim collects more properties of neighbors of poor and very poor vertices.

Claim 5.4. *Graph G possesses the following properties.*

(i) *If u is either a poor or a very poor vertex and uv is a lower link, a semi-link, or a sink, then $d_G(v) = d_{G^*}(v)$ and $d_G(u) = d_{G^*}(u)$.*

(ii) *If e is a sink, then $|N_G^2[e]| \leq 3\Delta$.*
If e is a lower link, then $|N_G^2[e] \setminus B(G)| \leq 3\Delta - 1$.
If e is a semi-link, then $|N_G^2[e] \setminus (B(G) \cup S(G))| \leq 3\Delta - 1$.

(iii) *If v is a sponsor of a very poor vertex u, then v is a Δ_{G^*}-vertex.*

(iv) *If u is a poor 3_{G^*}-vertex, then it is adjacent to at least one $4^+_{G^*}$-vertex.*

(v) *If w is a rival of a very poor vertex u, then all but at most one neighbor of w are pale.*

Proof. (i) Claim 5.3 (iii) and (iv) implies that $d_G(u) = d_{G^*}(u)$. If $d_{G^*}(v) \leq 3$, then Claim 5.3 (iv) implies $d_G(v) = d_{G^*}(v)$. Otherwise, u is either a very poor 2_{G^*}-vertex of type (T3) or a poor 3_{G^*}-vertex where v is a rival of a very poor vertex of type (T3). In any case, this implies $d_{G^*}(v) = 4$. Assume $N_{G^*}(v) = \{u, u', u'', z\}$ where each of u', u'' is either a very poor 2_{G^*}-vertex or a poor 3_{G^*}-vertex. Again Claim 5.3 (iii) and (iv) implies that $d_G(p) = d_{G^*}(p)$ for $p \in \{u, u', u''\}$. Then $d_G(z) + d_G(u) + d_G(u') + d_G(u'') \leq \Delta + 2 + 3 + 3 \leq 2\Delta + d_{G^*}(v)$, in other words, (5.1) holds. Thus Claim 5.3 (i) implies that $d_G(v) = d_{G^*}(v)$.

(ii) Assume $e = uv$ is a u-sink. This means u is a 3_{G^*}-vertex, v is a 2_{G^*}-vertex, and at least one vertex in $N_{G^*}(u) \setminus \{v\}$ has degree less than Δ in G. Then $d_G(u) = d_{G^*}(u) = 3$ by Claim 5.3 (iv). Thus

$$|N_G^2[e]| \leq \sum_{w \in N_{G^*}(u) \setminus \{v\}} d_G(w) + \sum_{w \in N_{G^*}(v) \setminus \{u\}} d_G(w) + |\{e\}| \leq 2\Delta - 1 + \Delta + 1 \leq 3\Delta.$$

Assume now $e = uv$ is a lower link of u. Then by definition and (i), $d_G(u) = 2$ and $2 \le d_G(v) \le 4$. If $d_G(v) = 4$, let t be the number of poor 3_{G^*}-vertices in $N_G(v)$; then $0 \le t \le 3$. Depending on the type (T1)–(T3) of v we have

$$|N_G^2[uv]| \le 1 + \sum_{y \in (N_G(u) \cup N_G(v)) \setminus \{u,v\}} d_G(y) \le \begin{cases} 2\Delta + 1 & \text{in case of (T1),} \\ 2\Delta + 3 & \text{in case of (T2),} \\ 2\Delta + 5 + t & \text{in case of (T3) with } t \le 2, \\ \Delta + 10 & \text{in case of (T3) with } t = 3. \end{cases}$$

Since $d_{G^*}(v) \le 4 < \Delta$, each poor 3_{G^*}-vertex w in $N_{G^*}(v)$ is incident to a sink. Thus

$$|N_G^2[uv] \setminus B(G)| \le \begin{cases} 2\Delta + 1 & \le 3\Delta - 1 \quad \text{in case of (T1),} \\ 2\Delta + 3 & \le 3\Delta - 1 \quad \text{in case of (T2),} \\ 2\Delta + 5 + t - t & \le 3\Delta - 1 \quad \text{in case of (T3).} \end{cases}$$

If $e = uv$ is a semi-link of v, then, by definition, v is the rival of some very poor vertex w and u is a poor 3_{G^*}-vertex. So by (i), $d_G(v) = d_{G^*}(v) = 4$. Then each poor 3_{G^*}-vertex $u' \in N_{G^*}(v)$ is incident to a sink, since $v \in N_{G^*}(u')$ satisfies $d_G(v) < \Delta$. Let t be the number of poor 3_{G^*}-vertices in $N_{G^*}(v)$.

$$|N_G^2[uv]| \le 1 + \sum_{y \in (N_G(u) \cup N_G(v)) \setminus \{u,v\}} d_G(y) \le 2\Delta + 2(3 - t) + 3t.$$

Since $N_G^2[uv]$ contains at least t sinks and $3 - t$ lower links, $|N_G^2[uv] \cap (B(G) \cup S(G))| \ge 3$. Hence $|N_G^2[ww'] \setminus (B(G) \cup S(G))| \le 2\Delta + 6 + t - 3 \le 3\Delta - 1$ since $\Delta \ge 7$ and $1 \le t \le 3$.

(iii) Suppose v is a sponsor of a very poor vertex u and $d_{G^*}(v) \le \Delta - 1$. Let w be the rival of u. Note that in each of the cases (T1)–(T3), $\Gamma_G(w) \setminus (S(G) \cup S'(G))$ contains at most one edge; let this edge be e'. Let H be the graph obtained from $G - uw$ by adding, if necessary, leaves adjacent only to v so that $d_H(v) = \Delta$. Then $d_H(u) = 1$ and since $d_{G^*}(v) \le \Delta - 1$, v has a 1_H-neighbor x distinct from u.

Adding leaves to a graph does not increase the maximum average degree, if it is at least 2. It also does a not create new 3-regular subgraph. So $\mathrm{Mad}(H) \le 3$ and H has no 3-regular subgraphs. Since H has fewer 2^+-vertices than G, it has a 3Δ-coloring f'. Since x and u are symmetric in H and $f'(vx) \ne f'(vu)$, we may assume that $f'(vu) \ne f'(e')$ by changing colors of vx and vu if necessary.

Let $f(e) := f'(e)$ for each edge $e \in E(H) \setminus (S(G) \cup S'(G) \cup B(G))$. Then $f|_{E(G)}$ is a partial 3Δ-coloring of G since $f'(vu) \ne f'(e')$. By (ii) and the fact that $uw \in S'(G) \cup S(G) \cup B(G)$, $(S'(G), S(G), B(G))$ is an $(f|_{E(G)}, 3\Delta)$-degenerate sequence for G. Thus we conclude that G is $(f|_{E(G)}, 3\Delta)$-degenerate. Thus Lemma 2.1 implies that $\chi_s'(G) \le 3\Delta$, a contradiction. This proves (iii).

(iv) Suppose that the neighbors of a poor 3_{G^*}-vertex u are v_1, v_2 and v_3, and $d_{G^*}(v_i) \le 3$ for $i \in [3]$. By the definition of a poor 3_{G^*}-vertex, we may assume that $d_{G^*}(v_1) = 2$ and $d_{G^*}(v_2) = d_{G^*}(v_3) = 3$. By Claim 5.3, $d_G(w) = d_{G^*}(w)$ for $w \in \{u, v_1, v_2, v_3\}$. Consider $H := G - u$, which has fewer vertices of degree at least 2 than G. The minimality of G implies that H has a 3Δ-coloring f. By the construction of H, f is a partial 3Δ-coloring of G. Note that $|N_G^2[uv_i]| \le 2\Delta + 6 \le 3\Delta$ for $i \in [3]$. Thus (uv_1, uv_2, uv_3) is an $(f, 3\Delta)$-degenerate sequence for G, and so G is $(f, 3\Delta)$-degenerate. So Lemma 2.1 implies that $\chi_s'(H) \le 3\Delta$, a contradiction. This proves (iv).

(v) By definition, a poor 3_{G^*}-vertex and a 2_{G^*}-vertex are pale. Since each neighbor of w possibly except one is either a poor 3_{G^*}-vertex or a 2_{G^*}-vertex, (v) follows. □

Claim 5.5. *No vertex is a sponsor of two distinct very poor vertices.*

Proof. Assume that a vertex v is a sponsor of distinct very poor vertices u and u'. By Claim 5.3(iii), both u and u' are 2_G-vertices. Let $N_{G^*}(u) = \{v, w\}$ and $N_{G^*}(u') = \{v, w'\}$. By Claim 5.4(iii), v is a Δ_{G^*}-vertex.

Case 1. $vw \in E(G)$ (this includes the case $w = u'$). Consider the graph $H := G - uw$. Since H is a proper subgraph of G, it satisfies the conditions of Theorem 5.1 and contains fewer 2^+-vertices than G. So by the minimality of G, the graph H has a 3Δ-coloring f. Since $vw \in E(G)$, f is a partial 3Δ-coloring of G, where only uw is not colored. By the definition of very poor vertices, $d_{G^*}(w) \le 4$ and v is the only possible neighbor of w that is neither poor nor very poor. Hence by Claims 5.3 and 5.4(i),

$$|N_G^2[uw]| \le 1 + d_G(v) + \sum_{x \in N_G(w) - \{v,u\}} d_G(x) \le 1 + \Delta + 2(3) < 3\Delta.$$

Thus we can extend f to uw, a contradiction.

Case 2. $vw \notin E(G)$ and $w = w'$. Consider the graph H obtained from $G - u$ by deleting the edge $u'w$ and adding the edge vw. Then H has fewer 2^+-vertices than G. Suppose $V(H)$ contains a set A with $\rho_H(A) < 0$. Since $\rho_G(A) \geq 0$, $wv \in E(H[A])$, so $w, v \in V(H)$. Also we may assume that $u' \notin A$, since $\rho_H(A - u') \leq \rho_H(A)$. However, since each of u and u' is adjacent to each of v and w, the graph $G[A \cup \{u, u'\}]$ has 4 more edges than $G[A]$. So

$$\rho_G(A \cup \{u, u'\}) = \rho_G(A) + 2(3) - 4(2) = (\rho_H(A) + 2) + 6 - 8 = \rho_H(A) < 0,$$

a contradiction. Thus $\rho_H(B) \geq 0$ for any $B \subseteq V(H)$. Similarly, if H contains a 3-regular subgraph H', then H' contains both v and w. This means w has two neighbors in G that are not pale. This contradicts Claim 5.4 (v). Thus Lemma 2.2 implies that H has no 3-regular subgraphs containing w. Hence H has no 3-regular subgraphs at all. So H satisfies the conditions of Theorem 5.1 and by the minimality of G, H has a 3Δ-coloring f'. Let

$$f(e) := \begin{cases} f'(e) & \text{if } e \in E(G) \cap E(H), \\ f'(vw) & \text{if } e = uv. \end{cases}$$

Since $vw \in E(H)$, colors $f(uv)$ and $f(u'v)$ are disjoint from $f(\Gamma_G(w))$. So $f|_{E(G)}$ is a partial 3Δ-coloring of G and the only non-colored edges are wu and wu'. Similarly to the end of the proof of Case 1, $d_{G^*}(w) \leq 4$ and at most one neighbor of w, is neither poor nor very poor. Hence by Claims 5.3 and 5.4(i), for $y \in \{u, u'\}$

$$|N_G^2[yw]| \leq 2 + d_G(v) + \sum_{x \in N_G(w) - \{u', u\}} d_G(x) \leq 2 + \Delta + \Delta + 3 < 3\Delta.$$

Thus we can extend f to uw and $u'w$, a contradiction.

Case 3. $|\{u, u', w, w'\}| = 4$ and $vw, vw' \notin E(G)$. Since every rival of a very poor vertex is incident to at most one edge that is not in $S(G) \cup S'(G)$, let e' be the unique edge incident to w' such that $e' \notin S(G) \cup S'(G)$, if it exists. Similarly to Case 2, consider the graph H_1 obtained from $G - u$ by deleting the edge $u'w'$ and adding the edge vw.

If H_1 has a 3Δ-coloring f_1, then we may assume that $\{f_1(vw), f_1(vu')\} = \{\alpha, \beta\}$ with $\alpha \neq \beta$ and $f_1(e') \neq \beta$. Let

$$f(e) := \begin{cases} f_1(e) & \text{if } e \in E(G) \setminus (S(G) \cup S'(G) \cup B(G) \cup \{uv, vu'\}) \\ \alpha & \text{if } e = uv \\ \beta & \text{if } e = vu'. \end{cases}$$

Then e' is the only edge in $\Gamma_{H_1}(w')$ that is colored by f. Also $\alpha, \beta \notin f(\Gamma_{H_1}(w))$, since (due to the edge vw) every edge in $\Gamma_{H_1}(w)$ is distance at most one from vu' and vw in H_1. Thus f is a partial 3Δ-coloring of G, and the uncolored edges are exactly the edges in $S(G) \cup S'(G) \cup B(G)$. (Note that $wu, w'u' \in S(G)$.) Hence Claim 5.4 (ii) implies that $(S'(G), S(G), B(G))$ is an $(f, 3\Delta)$-degenerate sequence for G; thus G is $(f, 3\Delta)$-degenerate, a contradiction.

Since H_1 has fewer 2^+-vertices than G, this means that H_1 does not satisfy the conditions of our theorem. This means that either there exists a set $A \subseteq V(H_1)$ with $\rho_{H_1}(A) < 0$ or H_1 has a 3-regular subgraph H_1'. If the latter holds, then H_1' must contain the edge wv since G has no 3-regular subgraphs. Then the neighbors of w in $H_1' - v$ are not pale in H_1 and hence in G, a contradiction to Claim 5.4 (v). We conclude that there exists a set $A_1 \subseteq V(H_1)$ with $\rho_{H_1}(A_1) < 0$. Since $\rho_G(A_1) \geq 0$, we know that $w, v \in A_1$ and $u' \notin A_1$. Then

$$\rho_G(A_1 \cup \{u\}) = \rho_G(A_1) + 3 - 2 \cdot 2 = \rho_{H_1}(A_1) + 2 + 3 - 4 = \rho_{H_1}(A_1) + 1 \geq 0.$$

It follows that $\rho_{H_1}(A_1) = -1$, $\rho_G(A_1 \cup \{u\}) = \rho_{H_1}(A_1) + 1 = 0$ and $\rho_G(A_1) = \rho_G(A_1 \cup \{u\}) - 3 + 2(2) = 1$. Also $w' \notin A_1$, since otherwise we have $\rho_G(A_1 \cup \{u, u'\}) = \rho_G(A_1) + 2 \cdot 3 - 4 \cdot 2 = -1$, a contradiction.

By symmetric argument, we can also find a set A_1' such that $w', v \in A_1'$, $u, w \notin A_1'$, $\rho_G(A_1' \cup \{u'\}) = 0$ and $\rho_G(A_1') = 1$. Then by Lemma 5.2,

$$\begin{aligned} \rho_G((A_1 \cup \{u\}) \cap (A_1' \cup \{u'\})) + \rho_G((A_1 \cup \{u\}) \cup (A_1' \cup \{u'\})) &\leq \rho_G(A_1 \cup \{u\}) + \rho_G(A_1' \cup \{u'\}) \\ &= 0 + 0 = 0. \end{aligned}$$

Since $(A_1 \cup \{u\}) \cap (A'_1 \cup \{u'\}) = A_1 \cap A'_1$, we conclude that

$$\rho_G(A_1 \cap A'_1) = 0. \tag{5.2}$$

If $ww' \in E(G)$, then by Lemma 5.2,

$$\begin{aligned}\rho_G(A_1 \cup A'_1 \cup \{u, u'\}) &= \rho_G(A_1 \cup \{u\}) + \rho_G(A'_1 \cup \{u'\}) - \rho_G((A_1 \cup \{u\}) \cap (A'_1 \cup \{u'\})) \\ &\quad - 2|E_G(\{u\} \cup A_1 \setminus A'_1, \{u'\} \cup A'_1 \setminus A_1)| \\ &\le 0 + 0 - 0 - 2(1) = -2,\end{aligned}$$

a contradiction. Thus $ww' \notin E(G)$.

Now we consider the graph H_2 obtained from $G - \{wu, w'u'\}$ by adding the edge ww'. Recall that e' is the unique edge incident to w' such that $e' \notin S(G) \cup S'(G)$, if it exists. Let e'' be the unique edge incident to w such that $e'' \notin S(G) \cup S'(G)$, if it exists.

Assume H_2 has a 3Δ-coloring f_2. Then $f_2(uv) \ne f_2(u'v)$. Since e' and e'' are distance one from each other in $H_2, f_2(e') \ne f_2(e'')$. We may assume that $f_2(e') \ne f_2(u'v)$ and $f_2(e'') \ne f_2(uv)$ by switching the colors of uv and $u'v$ if necessary.

Then we let

$$f(e) = f_2(e) \text{ for } e \in E(G) \setminus (S(G) \cup S'(G) \cup B(G)).$$

The only edge in $N^2_G(uv)$ incident to w and colored in f_2 is e'', and the only edge in $N^2_G(u'v)$ incident to w' colored in f_2 is e'. Since $f_2(e'') \ne f_2(uv)$ and $f_2(e') \ne f_2(u'v)$, f is a partial 3Δ-coloring of G. Now Claim 5.4 (ii) implies that $(S'(G), S(G), B(G))$ is an $(f, 3\Delta)$-degenerate sequence for G. Thus G is $(f, 3\Delta)$-degenerate, a contradiction. So H_2 must not have a 3Δ-coloring.

Since H_2 contains fewer 2^+-vertices than G, this means that H_2 does not satisfy the conditions of our theorem. So either there exists a set $A_2 \subseteq V(H_2)$ with $\rho_{H_2}(A_2) < 0$ or H_2 has a 3-regular subgraph H'_2. In the latter case, H'_2 must contain the edge ww', since G contains no 3-regular subgraphs. Then the neighbors of w in $H'_2 - w'$ are not pale in H_2 and hence in G, a contradiction to Claim 5.4 (v). We conclude that there exists a set $A_2 \subseteq V(H_2)$ with $\rho_{H_2}(A_2) < 0$. Since $\rho_G(A_2) \ge 0$, we know that $w, w' \in A_2$. We may also assume that $u, u' \notin A_2$ since $\rho_{H_2}(A_2 \setminus \{u, u'\}) \le \rho_{H_2}(A_2)$. Then $\rho_G(A_2) = \rho_{H_2}(A_2) + 2 \le 1$. Also $v \notin A_2$ since otherwise $\rho_G(A_2 \cup \{u, u'\}) = \rho_G(A_2) + 2 \cdot 3 - 4 \cdot 2 \le -1$, a contradiction. Thus we have a set A_2 with $\rho_{H_2}(A_2) \le 1$, $w, w' \in A_2$ and $u, v, u' \notin A_2$.

Since $\{v, w, w'\} \subseteq A_1 \cup A_2$ and $\{u, u'\} \cap (A_1 \cup A_2) = \emptyset$, $\rho_G(A_1 \cup A_2 \cup \{u, u'\}) = \rho_G(A_1 \cup A_2) + 2(3) - 4(2) = \rho_G(A_1 \cup A_2) - 2$. It follows that $\rho_G(A_1 \cup A_2) \ge 2$. So by Lemma 5.2,

$$\rho_G(A_1 \cap A_2) = \rho_G(A_1) + \rho_G(A_2) - \rho_G(A_1 \cup A_2) \le 1 + 1 - 2 = 0.$$

Then by (5.2) and again by Lemma 5.2

$$\rho_G((A_1 \cap A'_1) \cup (A_1 \cap A_2)) \le \rho_G(A_1 \cap A'_1) + \rho_G(A_1 \cap A_2) = 0 + 0 = 0.$$

Yet, $(A_1 \cap A'_1) \cup (A_1 \cap A_2)$ contains v and w and does not contain u. So

$$\rho_G((A_1 \cap A'_1) \cup (A_1 \cap A_2) \cup \{u\}) = \rho_G((A_1 \cap A'_1) \cup (A_1 \cap A_2)) + 3 - 2(2) = -1,$$

a contradiction to the choice of G. This proves the lemma. □

Claim 5.6. *Let $v \in V(G^*)$.*

(i) *If $d_{G^*}(v) < \Delta$, then at least one vertex in $N_{G^*}(v)$ is neither poor nor very poor.*
(ii) *If $d_{G^*}(v) = \Delta$, then either at least one vertex in $N_{G^*}(v)$ is neither poor nor very poor or v is not a sponsor of a very poor vertex.*

Proof. Let $d_{G^*}(v) = s$ and $N_{G^*}(v) = \{u_1, \ldots, u_s\}$ where each u_i is either poor or very poor. Further assume that $u_1, \ldots, u_t$ $(t \le s)$ are the poor 3_{G^*}-vertices in $N_{G^*}(v)$. By Claim 5.3,

$$d_{G^*}(x) = d_G(x) \text{ for each } x \in N_{G^*}[v]. \tag{5.3}$$

In particular, $d_{G^*}(v) = d_G(v) = s$. For $i \in [t]$, since u_i is a poor 3_{G^*}-vertex, let $N_G(u_i) = \{v, u_i', u_i''\}$, where u_i'' is the unique 2_{G^*}-vertex in $N_G(u_i)$, $N_G(u_i'') = \{u_i, x_i\}$, and let $e_i := u_i u_i''$. Note that e_i may or may not be a sink. For $i \in [s] \setminus [t]$, we know $d(u_i) = 2$, so let $N_G(u_i) = \{v, u_i'\}$.

(i) Suppose $s < \Delta$. Recall that $d_G(v) = s < \Delta$. For each $i \in [t]$, the poor 3_{G^*}-vertex u_i is adjacent to the $(\Delta - 1)_G^-$-vertex v and hence is incident to exactly one sink e_i.

Let $H := G - v$. By the minimality of G, the graph H has a 3Δ-coloring f'. Let

$$f(e) := f'(e) \text{ for } e \in E(G) \setminus B(G).$$

Then f is a partial 3Δ-coloring of G. Consider the ordering $(vu_1, \dots, vu_s, B(H))$ of the edges not colored by f. First, since $d_G(v) \le \Delta - 1$, for each $i \in [t]$,

$$\begin{aligned}|N_G^2[vu_i] \setminus (B(G) \cup \{vu_{i+1}, \dots, vu_s\})| &\le d_G(u_i') + d_G(u_i'') + \sum_{j \ne i} |N_G(u_j) \setminus B(G)| + |\{vu_i\}| \\ &\le \Delta + 2 + 2(d_G(v) - 1) + 1 \le 3\Delta - 1.\end{aligned}$$

Similarly, if $t < i \le s$, then

$$\begin{aligned}|N_G^2[vu_i] \setminus (B(G) \cup \{vu_{i+1}, \dots, vu_s\})| &\le d_G(u_i') + \sum_{j \ne i} |N_G(u_j) \setminus B(G)| + |\{vu_i\}| \\ &\le \Delta + 2(d_G(v) - 1) + 1 \le 3\Delta - 3.\end{aligned}$$

Claim 5.4 (ii) implies that $|N_G^2[e]| \le 3\Delta$ for each $e \in B(G)$. Therefore, $(vu_1, \dots, vu_s, B(G))$ is an $(f, 3\Delta)$-degenerate sequence for G. Hence G is $(f, 3\Delta)$-degenerate, a contradiction to Lemma 2.1 and the assumption that G does not have a 3Δ-coloring.

(ii) Assume $s = \Delta$ and u_Δ is very poor, so that $t < s$. Let u' be the rival of u_Δ and let e' be the unique edge incident to u' where $e' \notin S(G) \cup S'(G)$, if it exists.

Consider $H := G - v$. By the minimality of G, graph H has a 3Δ-coloring f'. Let

$$f(e) := f'(e) \text{ for } e \in E(G) \setminus (\{e_1, \dots, e_t\} \cup S(G) \cup S'(G) \cup B(G)).$$

Then f is a partial 3Δ-coloring of G since $N_H^2[e] = N_G^2[e] \setminus \Gamma_G(v)$ for each $e \in E(H)$. Let $B'(G) := B(G) \setminus \{e_1, \dots, e_t\}$ and $P(G) := S'(G) \cup S(G) \cup B'(G)$.

Consider the sequence $(vu_1, \dots, vu_{\Delta-1}, e_1, \dots, e_t, vu_\Delta, S'(G), S(G), B'(G))$ of edges not colored by f. We want to show that it is an $(f, 3\Delta)$-degenerate sequence for G. Note that no edge incident to u_Δ is colored by f. First, for $i \in [t]$,

$$\begin{aligned}|N_G^2[vu_i] \setminus (\{e_1, \dots, e_t, vu_\Delta\} \cup P(G))| &\le d_G(u_i') + (d_G(u_i'') - 1) + \sum_{j \notin \{i, \Delta\}} |N_G(u_j) \setminus \{e_j\}| + |\{vu_i\}| \\ &\le \Delta + 1 + 2(\Delta - 2) + 1 \le 3\Delta - 2.\end{aligned}$$

If $t < i \le \Delta - 1$, then

$$\begin{aligned}|N_G^2[vu_i] \setminus (\{e_1, \dots, e_t, vu_\Delta\} \cup P(G))| &\le d_G(u_i') + \sum_{j \notin \{i, \Delta\}} |N_G(u_j) \setminus \{e_1, \dots, e_t\}| + |\{vu_i\}| \\ &\le \Delta + 2(\Delta - 2) + 1 \le 3\Delta - 3.\end{aligned}$$

For $i \in [t]$,

$$\begin{aligned}|N_G^2[e_i] \setminus (\{vu_\Delta\} \cup P(G))| &\le d_G(x_i) + d_G(u_i') + |N_G(v) \setminus \{vu_\Delta\}| + |\{e_i\}| \\ &\le \Delta + \Delta + \Delta - 1 + 1 \le 3\Delta.\end{aligned}$$

Also

$$|N_G^2[vu_\Delta] \setminus P(G)| \le \sum_{i=1}^{\Delta-1} d_G(u_i) + |\{e'\}| + |\{vu_\Delta\}| \le 3(\Delta - 1) + 2 \le 3\Delta - 1.$$

These inequalities together with Claim 5.4 (iii) imply that the sequence $(vu_1, \dots, vu_{\Delta-1}, e_1, \dots, e_t, vu, S'(G), S(G), B'(G))$ is an $(f, 3\Delta)$-degenerate sequence for G. So, G is $(f, 3\Delta)$-degenerate, a contradiction to Lemma 2.1 and the choice of G. This proves the claim. □

Claim 5.7. *Suppose v is a poor 3_{G^*}-vertex and $N_{G^*}(v) = \{v_1, v_2, v_3\}$, where $d_{G^*}(v_1) = 4$, $d_{G^*}(v_2) = 3$, and $d_{G^*}(v_3) = 2$. Then v_1 has at least two neighbors in G^* where each of them is neither poor nor very poor.*

Proof. Suppose that under the conditions of the lemma, $N_{G^*}(v_1) = \{v, u_1, u_2, x\}$, where each of u_1 and u_2 is either poor or very poor. Let $N_{G^*}(v_2) = \{v, y_1, y_2\}$ and $N_{G^*}(v_3) = \{v, z\}$. Since v, u_1, and u_2 are all poor or very poor, their degrees in G are the same as in G^*. Since $\Delta \geq 7$,

$$2\Delta + d_{G^*}(v_1) = 2\Delta + 4 > 10 + \Delta > 3 + 3 + 3 + \Delta \geq d_G(v) + d_G(u_1) + d_G(u_2) + d_G(x).$$

Thus Claim 5.3 (i) implies that $d_G(v_1) = d_{G^*}(v_1) = 4$. Also by Claim 5.3 (iv), $d_G(v_2) = d_{G^*}(v_2) = 3$.

Consider $H := G - v$. By the minimality of G, the graph H has a 3Δ-coloring f. By the construction of H, f is a partial 3Δ-coloring of G.

Since $\Delta \geq 7$,

$$\begin{aligned}
|N_G^2[vv_1] \setminus \{vv_2, vv_3\}| &\leq d_G(x) + (d_G(v_2) - 1) + (d_G(v_3) - 1) + \sum_{i=1}^{2} d_G(u_i) + |\{vv_1\}| \\
&\leq \Delta + 2 + 1 + 2 \cdot 3 + 1 \leq 3\Delta, \\
|N_G^2[vv_2] \setminus \{vv_3\}| &\leq d_G(y_1) + d_G(y_2) + d_G(v_1) + d_G(v_3) - 1 + |\{vv_2\}| \\
&\leq 2\Delta + 4 + 1 + 1 \leq 3\Delta, \\
|N_G^2[vv_3]| &\leq d_G(z) + d_G(v_2) + d_G(v_1) + |\{vv_3\}| \leq \Delta + 3 + 4 + 1 \leq 3\Delta.
\end{aligned}$$

Thus (vv_1, vv_2, vv_3) is an $(f, 3\Delta)$-degenerate sequence for G. So G is $(f, 3\Delta)$-degenerate, a contradiction to the choice of G. $\square$

5.3. Discharging

Since G^* is a subgraph of G, we know $\mathrm{Mad}(G^*) \leq 3$ and G^* does not contain 3-regular subgraphs. For every $v \in V(G^*)$, define *the initial charge* $\mathrm{ch}(v) := d_{G^*}(v)$. Since $\mathrm{Mad}(G^*) \leq 3$, we have $\sum_{v \in V(G^*)} \mathrm{ch}(v) \leq 3|V(G^*)|$. We will move the charge among vertices without changing the total sum of charge according to the discharging rules below.

(R1) If a Δ_{G^*}-vertex v is a sponsor of a very poor vertex u, then v gives charge 1 to u.
(R2) If $5 \leq d_{G^*}(v) \leq \Delta$ and u is a poor vertex in $N_{G^*}(v)$, then v gives charge $1/2$ to u.
(R3) If a 4_{G^*}-vertex v is adjacent to a poor 2_{G^*}-vertex u, then v gives charge $1/2$ to u.
(R4) If a 4_{G^*}-vertex v is adjacent to a poor 3_{G^*}-vertex u, then we do one of the following:

- (R4A) If $N_{G^*}(v)$ contain at least two vertices that are neither poor nor very poor, then v gives charge $1/2$ to u.
- (R4B) Otherwise, v gives charge $1/4$ to u.

(R5) If a poor 3_{G^*}-vertex v is adjacent to a poor 2_{G^*}-vertex u, then v gives charge $1/2$ to u.

For every $v \in V(G^*)$, let $\mathrm{ch}^*(v)$ be the *final charge* of v, which is the charge of v after the distribution. Since the total sum of charge did not change,

$$\sum_{v \in V(G^*)} \mathrm{ch}^*(v) = \sum_{v \in V(G^*)} \mathrm{ch}(v) \leq 3|V(G^*)|. \tag{5.4}$$

See Fig. 2 for an illustration of the discharging rules.

The next claim shows important properties of the final charge ch^*.

Claim 5.8. *Let $v \in V(G^*)$.*

(i) *If v is a 2_{G^*}-vertex, then $\mathrm{ch}^*(v) = 3$.*
(ii) *If v is a 3_{G^*}-vertex, then $\mathrm{ch}^*(v) \geq 3$.*

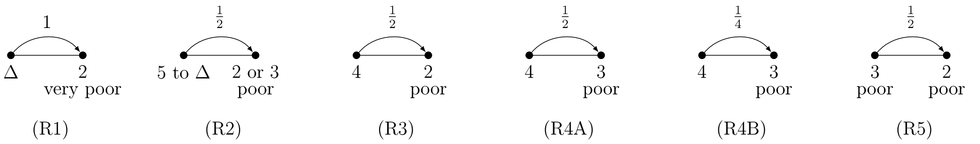

Fig. 2. The discharging rules.

(iii) *If v is a 4_{G^*}-vertex, then* $\mathrm{ch}^*(v) \geq 3$. *Moreover, if* $\mathrm{ch}^*(v) = 3$, *then v is adjacent to exactly two poor vertices and no very poor vertices.*
(iv) *If* $5 \leq d_{G^*}(v) \leq \Delta - 1$, *then* $\mathrm{ch}^*(v) \geq 3$. *Moreover, if* $\mathrm{ch}^*(v) = 3$, *then* $d_{G^*}(v) = 5$ *and v has exactly four poor neighbors.*
(v) *If v is a Δ_{G^*}-vertex, then* $\mathrm{ch}^*(v) > 3$.

Proof. (i) Suppose $d_{G^*}(v) = 2$. Then v is either poor or very poor. If v is very poor, then it is adjacent to its sponsor w, and by Claim 5.4 (iii), $d_{G^*}(w) = \Delta$. Hence by (R1), w gives charge 1 to v, so $\mathrm{ch}^*(v) = 3$.

If v is poor but not very poor, then both neighbors of v are $3^+_{G^*}$-vertices, and each 3_{G^*}-neighbor of v is poor. Thus, by rules (R2), (R3), or (R5), v receives $1/2$ from each of its two neighbors in G^*. So $\mathrm{ch}^*(v) = 3$.

(ii) Suppose $d_{G^*}(v) = 3$. If v is not poor, then v does not send or receive any charge, so $\mathrm{ch}^*(v) = \mathrm{ch}(v) = 3$. Now assume v is poor. Then by Claim 5.4 (iv), v has a $4^+_{G^*}$-neighbor. By (R5), v gives $1/2$ to its unique 2_{G^*}-neighbor, and by (R2) or (R4) receives at least $1/4$ from each $4^+_{G^*}$-neighbor. Thus, if $\mathrm{ch}^*(v) < 3$, then v has only one $4^+_{G^*}$-neighbor, say u. Furthermore, by (R2), $d_{G^*}(u) = 4$, and by (R4), u has at most one neighbor in G^* that is neither poor nor very poor. But this contradicts Claim 5.7.

(iii) Suppose $d_{G^*}(v) = 4$. If v is the rival of a very poor vertex u, then by the definition (T3), every 2_{G^*}-neighbor of v is very poor and hence v does not give to its 2_{G^*}-neighbors anything. In this case, it either gives $1/2$ to its unique 3_{G^*}-neighbor by (R4A) or gives $1/4$ to each of its at most three poor neighbors by (R4B). In both cases, $\mathrm{ch}^*(v) \geq 4 - 3 \cdot \frac{1}{4} > 3$.

Otherwise, v is not adjacent to any very poor vertices. Then by Claim 5.6 (i), v is adjacent to at most three poor vertices. If v has three poor neighbors, then all of them are 3_{G^*}-vertices, because in this case each 2_{G^*}-neighbor of v is very poor. Thus v sends charge $1/4$ to its three poor neighbors by (R4B), so $\mathrm{ch}^*(v) \geq 4 - 3 \cdot \frac{1}{4} > 3$. The last possibility is that v has at most two poor neighbors. Since v gives to any poor neighbor at most $1/2$, the only possibility to have $\mathrm{ch}^*(v) \leq 3$ is that v has exactly two poor neighbors and gives $1/2$ to each of them, in which case $\mathrm{ch}^*(v) = 3$. This proves (iii).

(iv) Suppose $5 \leq d_{G^*}(v) \leq \Delta - 1$. By (R2), v gives charge $1/2$ to each of its poor neighbors. By Claim 5.6 (i), the number of such neighbors is at most $d_{G^*}(v) - 1$. So

$$\mathrm{ch}^*(v) \geq \mathrm{ch}(v) - \frac{1}{2}(d_{G^*}(v) - 1) = \frac{d_{G^*}(v) + 1}{2} \geq 3.$$

Moreover, if $\mathrm{ch}^*(v) = 3$, then $d_{G^*}(v) = 5$ and v is adjacent to exactly four poor neighbors.

(v) Suppose $d_{G^*}(v) = \Delta$. If v has no very poor neighbors, then by (R2) it gives at most $1/2$ to each of its neighbors, and $\mathrm{ch}^*(v) \geq \mathrm{ch}(v) - \frac{d_{G^*}(v)}{2} = \frac{d_{G^*}(v)}{2} \geq \frac{\Delta}{2} > 3$. Otherwise, by Claim 5.5, it has only one very poor neighbor (to which it gives 1 by (R1)), but then by Claim 5.6 (ii), it has a neighbor that is neither poor nor very poor. Thus by (R1) and (R2), again

$$\mathrm{ch}^*(v) \geq d_{G^*}(v) - 1 - \frac{\Delta - 2}{2} = \Delta/2 > 3. \qquad \square$$

Claim 5.8 implies that $\sum_{v \in V(G^*)} \mathrm{ch}^*(v) \geq 3|V(G^*)|$ and together with (5.4) yields

$$\mathrm{ch}^*(v) = 3 \text{ for each } v \in V(G^*). \tag{5.5}$$

This yields the following facts.

Claim 5.9. *Graph G has the following properties.*

(i) *$\Delta(G^*) \le 5$.*
(ii) *G^* has no very poor vertices.*
(iii) *$B(G) = \emptyset$.*
(iv) *If v is a 5_{G^*}-vertex, then $d_G(v) = d_{G^*}(v)$.*
(v) *There are no 5_{G^*}-vertices; in other words, $\Delta(G^*) \le 4$.*
(vi) *There are no poor 3_{G^*}-vertices.*

Proof. (i) Claim (i) follows from (5.5) and Claim 5.8 (parts (iv) and (v)).

(ii) If v is a very poor vertex in G^*, then by Claim 5.4 (iii), it has a Δ_{G^*}-neighbor, contradicting (i).

(iii) Suppose that G^* has a u-sink uw, which means that $d_{G^*}(w) = 2$, $d_{G^*}(u) = 3$, and

$$d_G(u') + d_G(u'') \le 2\Delta - 1 \tag{5.6}$$

where $N_{G^*}(u) = \{w, u', u''\}$ and $N_{G^*}(w) = \{u, w'\}$. Recall that by Claim 5.3, $d_G(w) = d_{G^*}(w) = 2$ and $d_G(u) = d_{G^*}(u) = 3$. Let H be obtained from $G - uw$ by adding leaves adjacent only to w' so that the degree of w' in H is Δ. Since $d_H(w) = 1$, H has fewer 2^+-vertices than G. Also, $\mathrm{Mad}(H) \le 3$ and H has no 3-regular subgraphs. So by the minimality of G, the graph H has a 3Δ-coloring f. By (i) and the fact that $d_H(w) = 1$, vertex w' has at least $\Delta - 4 \ge 3$ 1_H-neighbors, including w. Hence we can switch the colors of the pendant edges incident to w' so that $f(w'w) \notin \{f(uu'), f(uu'')\}$. Then $f|_{E(G)}$ is a partial 3Δ-coloring of G, where the only non-colored edge is uw. But by (5.6),

$$|N_G^2[uw]| \le d_G(u') + d_G(u'') + d_G(w') + 1 \le (2\Delta - 1) + \Delta + 1 = 3\Delta,$$

so we can extend f to uw, a contradiction to the choice of G. This proves (iii).

(iv) Suppose $d_{G^*}(v) = 5$. By Claim 5.8 (iv), we may assume that $N_{G^*}(v) = \{u_1, \ldots, u_5\}$, where each of u_1, u_2, u_3, u_4 is either poor or very poor. In particular, by Claim 5.3, $d_G(u_i) = d_{G^*}(u_i) \le 3$ for $i \in [4]$. Thus,

$$\sum_{i=1}^{5} d_G(u_i) \le 3(4) + \Delta \le 5 + 2\Delta = d_{G^*}(v) + 2\Delta,$$

and Claim 5.3 (i) implies $d_G(v) = d_{G^*}(v)$.

(v) Suppose $d_{G^*}(v) = 5$ and $N_{G^*}(v) = \{u_1, \ldots, u_5\}$, where each of u_1, u_2, u_3, u_4 is either poor or very poor. If some u_i is a poor 3_{G^*}-vertex adjacent to a 2_{G^*}-vertex w_i, then by (iv), u_iw_i is the u_i-sink. But this contradicts (iii). Thus

$$\text{for each } i \in [4],\ u_i \text{ is a poor } 2_{G^*}\text{-vertex.} \tag{5.7}$$

So we may assume that $N_G(u_i) = \{v, w_i\}$ for $i \in [4]$. Let H be obtained from $G - vu_1$ by adding leaves adjacent only to w_1 so that the degree of w_1 in H is Δ. Since $d_H(u_1) = 1$, H has fewer 2^+-vertices than G. Also, $\mathrm{Mad}(H) \le 3$ and H has no 3-regular subgraphs. So by the minimality of G, the graph H has a 3Δ-coloring f. By (i) and the fact that $d_H(u_1) = 1$, the number of 1_H-neighbors of the vertex w_1, including u_1, is at least $\Delta - 4 \ge 3$. Hence we can switch the colors of the pendant edges incident to w_1 so that $f(w_1u_1) \ne f(vu_5)$.

If $f(w_1u_1) \notin \{f(vu_2), f(vu_3), f(vu_4)\}$, then $f|_{E(G)}$ is a partial 3Δ-coloring of G, where the only uncolored edge is u_1v. But in this case by (5.7),

$$|N_G^2[u_1v]| \le d_G(w_1) + \sum_{i=2}^{5} d_G(u_i) + 1 \le \Delta + 3(2) + \Delta + 1 = 2\Delta + 7 \le 3\Delta, \tag{5.8}$$

so we can extend f to u_1v, a contradiction to the choice of G.

Thus by symmetry, we may assume $f(w_1u_1) = f(vu_2)$. Then the coloring f' obtained from $f|_{E(G)}$ by uncoloring vu_2 is a partial 3Δ-coloring of G, where the only uncolored edges are u_1v and u_2v. Again by (5.8), we can extend f' to u_1v and then by the similar inequality for $|N_G^2[u_2v]|$, extend it to u_2v, a contradiction to the choice of G. This proves (v).

(vi) Suppose that v is a poor 3_{G^*}-vertex with $N_{G^*}(v) = \{u_1, u_2, u_3\}$ where $d_{G^*}(u_1) = 2$ and $d_{G^*}(u_2), d_{G^*}(u_3) \ge 3$. By (v), $d_{G^*}(u_2), d_{G^*}(u_3) \le 4$. Moreover, if, say, $d_{G^*}(u_2) = 3$, then by Claim 5.3 (iv),

$d_G(u_2) = 3$. Hence vu_1 is a v-sink, yet, this contradicts (iii). Therefore $d_{G^*}(u_2) = d_{G^*}(u_3) = 4$. So, by Claim 5.8 (iii), each of u_2 and u_3 has exactly two poor neighbors in G^*. Now by (R4), each of u_2 and u_3 gives 1/2 to v, while v gives to u_1 only 1/2 by (R5). Hence $\mathrm{ch}^*(v) = 3 + 2 \cdot \frac{1}{2} - 1/2 = 7/2 > 3$, a contradiction to (5.5). $\square$

Claim 5.9 implies that the degree of a vertex in G^* must be in $\{2, 3, 4\}$. Moreover, since G^* has neither very poor vertices nor poor 3_{G^*}-vertices and each 4_{G^*}-vertex is adjacent to exactly two poor vertices,

$$\begin{aligned}&\textit{each } 2_{G^*}\textit{-vertex is adjacent only to } 4_{G^*}\textit{-vertices, and each } 4_{G^*}\textit{-vertex is}\\ &\quad\textit{adjacent to exactly two } 2_{G^*}\textit{-vertices.}\end{aligned} \tag{5.9}$$

The last claim that we need is:

Claim 5.10. *If v is a 4_{G^*}-vertex, then $d_G(v) = d_{G^*}(v)$.*

Proof. Let v be a 4_{G^*}-vertex. By (5.9), we may assume that $N_{G^*}(v) = \{u_1, u_2, u_3, u_4\}$, where $d_G(u_1) = d_G(u_2) = 2$. So

$$\sum_{i=1}^{4} d_G(u_i) \leq 2(2) + 2\Delta = 2\Delta + d_{G^*}(v).$$

Hence by Claim 5.3 (i), $d_G(v) = d_{G^*}(v) = 4$. $\square$

Now we are ready to finish the proof of Theorem 5.1. Since $\mathrm{Mad}(G^*) \leq 3$ and G^* has no 3-regular subgraphs, it has a vertex u with $d_G(u) = d_{G^*}(u) = 2$. Let $N_G(u) = \{v, w\}$. By (5.9), $d_{G^*}(v) = d_{G^*}(w) = 4$. So by Claim 5.10, $d_G(v) = d_G(w) = 4$. Let $N_G(v) = \{u, v_1, v_2, v_3\}$. By (5.9), we may assume that $d_G(v_1) = 2$.

Let H be the graph obtained from $G - uv$ by adding $\Delta - 4$ leaves adjacent only to w so that $d_H(w) = \Delta$. Since $d_H(u) = 1$, H has fewer 2^+-vertices than G. Also, $\mathrm{Mad}(H) \leq 3$ and H has no 3-regular subgraphs. So by the minimality of G, the graph H has a 3Δ-coloring f. Since $d_G(w) = 4$, the number of 1_H-neighbors of the vertex w, including u, is at least $\Delta - 3 \geq 4$. Hence we can switch the colors of the pendant edges incident to w so that $f(wu) \notin \{f(vv_1), f(vv_2), f(vv_3)\}$. Thus $f|_{E(G)}$ is a partial 3Δ-coloring of G, where the only uncolored edge is uv. However,

$$|N_G^2[uv]| \leq d_G(w) + \sum_{i=1}^{3} d_G(v_i) + 1 \leq 4 + 2 + 2\Delta + 1 = 2\Delta + 7 \leq 3\Delta,$$

and so we can extend f to uv, a contradiction to the choice of G. This finishes the proof of Theorem 5.1.

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