On a theorem by Erdős, Rubin, and Taylor

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Abstract

Erdős, Rubin, and Taylor [5, p. 129] found a nice correspondence between choosability of a complete bipartite graph of a given order and 2-colorability (in ordinary sense) of uniform hypergraphs with few edges. In this note we use their ideas to derive similar correspondences for complete k-partite graphs and complete k-uniform k-partite hypergraphs.

1 Introduction

Let $m(r,k)$ denote the minimum number of edges in an r-uniform hypergraph with chromatic number greater than k and $N(k,r)$ denote the minimum number of vertices in a k-partite graph with list chromatic number greater than r.

Erdős, Rubin, and Taylor [5, p. 129], proved the following correspondence between $m(r,2)$ and $N(2,r)$.

Theorem 1 *For every $r \geq 2$, $m(r,2) \leq N(2,r) \leq 2m(r,2)$.*

This nice result shows close relations between ordinary hypergraph 2-coloring and list coloring of complete bipartite graphs. Note that $m(r,2)$ was studied in [2, 3, 4, 9]. Using known bounds on $m(r,2)$, Theorem 1 yields the corresponding bounds for $N(2,r)$:

$$c\ 2^r\sqrt{\frac{r}{\ln r}} \leq N(2,r) \leq C\ 2^r r^2.$$

Theorem 1 can be extended in a natural way in two directions: to complete k-partite graphs and to k-uniform k-partite hypergraphs. In this note we present these extensions (using the ideas of Erdős, Rubin, and Taylor).

A vertex t-coloring of a hypergraph G is *panchromatic* if every of t colors is used on every edge of G. Thus, an ordinary 2-coloring is panchromatic. Some results on the existence of panchromatic colorings for hypergraphs with few edges could be found in [7]. Let $p(r,k)$ denote the minimum number of edges in an r-uniform hypergraph not admitting any panchromatic k-coloring. Note that $p(r,2) = m(r,2)$. The first extension of Theorem 1 is the following.

*This work was partially supported by the NSF grant DMS-0099608.

Theorem 2 *For every $r \geq 2$ and $k \geq 2$, $p(r,k) \leq N(k,r) \leq k\ p(r,k)$.*

It follows from Alon's results in [1] that for some $0 < c_1 < c_2$ and every $r \geq 2$ and $k \geq 2$,

$$\exp\{c_1 r/k\} \leq N(k,r) \leq k \exp\{c_2 r/k\}.$$

Therefore, by Theorem 2 we get reasonable bounds on $p(r,k)$ for fixed k and large r:

$$\exp\{c_1 r/k\}/k \leq p(r,k) \leq k \exp\{c_2 r/k\}.$$

We say that a k-uniform hypergraph G is *k-partite*, if $V(G)$ can be partitioned into k sets so that every edge contains exactly one vertex from every part. Let $Q(k,r)$ denote the minimum number of vertices in a k-uniform k-partite hypergraph with list chromatic number greater than r.

Theorem 3 *For every $r \geq 2$ and $k \geq 2$, $m(r,k) \leq Q(k,r) \leq k\ m(r,k)$.*

From [4] and [6] we know that

$$c_1 k^r \left(\frac{r}{\ln r}\right)^{1-1/\lfloor 1+\log_2 k\rfloor} \leq m(r,k) \leq c_2 k^r r^2 \log k.$$

Thus, Theorem 3 yields that

$$c_1 k^r \left(\frac{r}{\ln r}\right)^{1-1/\lfloor 1+\log_2 k\rfloor} \leq Q(k,r) \leq c_2 k^{r+1} r^2 \log k.$$

2 Proof of Theorem 2

Let $G = (V,E)$ be an r-uniform hypergraph not admitting any panchromatic k-coloring with $E = \{e_1, \ldots, e_{p(r,k)}\}$. Consider the complete k-partite graph $H = (W,A)$ with parts $W_1, \ldots, W_k$ and $W_i = \{w_{i,1}, \ldots, w_{i,|E|}\}$ for $i = 1, \ldots, k$. The ground set for lists will be V. Recall that every e_i is an r-subset of V. For every $i = 1, \ldots, k$ and $j = 1, \ldots, |E|$, assign $w_{i,j}$ the list $L(w_{i,j}) = e_j$.

Assume that H has a coloring f from the lists. Then f produces a k-coloring g_f of V as follows: we let $g_f(v)$ be equal to the index i such that $v = f(w_{i,j})$ for some j or be equal to 1 if there is no such $w_{i,j}$ at all. Since H is a complete k-partite graph, this rule defines i uniquely. Since for every j all vertices in $\{w_{1,j}, w_{2,j}, \ldots, w_{k,j}\}$ must get different colors, g_f is a panchromatic k-coloring of G, a contradiction. This proves that $N(k,r) \leq k\ p(r,k)$.

Now, consider a complete k-partite graph $H = (W,A)$ with parts $W_1, \ldots, W_k$ and $|W| < p(r,k)$. Let L be an arbitrary r-uniform list for W. Let $G = (V,E)$ be the hypergraph with $V = \bigcup_{w \in W} L(w)$ and $E = \{L(w) \mid w \in W\}$. Since $|E| = |W| < p(r,k)$, there exists a panchromatic k-coloring g of G. Define the coloring f_g of W as follows: if $w \in W_i$, choose in the edge $L(w)$ of G any vertex v with $g(v) = i$ and let $f_g(w) = v$. Then vertices in different W_i cannot get the same color, and f is a coloring from the lists of vertices in H. This proves that $N(k,r) \geq p(r,k)$.

3 Proof of Theorem 3

Let $G = (V, E)$ be an r-uniform hypergraph not admitting any k-coloring with $E = \{e_1, \dots, e_{m(r,k)}\}$. Consider the complete k-partite k-uniform hypergraph $H = (W, A)$ with parts $W_1, \dots, W_k$ and $W_i = \{w_{i,1}, \dots, w_{i,|E|}\}$ for $i = 1, \dots, k$. The ground set for lists will be V. Recall that every e_i is an r-subset of V. For every $i = 1, \dots, k$ and $j = 1, \dots, |E|$, assign $w_{i,j}$ the list $L(w_{i,j}) = e_j$.

Assume that H has a coloring f from the lists. Note that no color v is present on every W_i, since otherwise H would have an edge with all vertices of color v. Thus, f produces a k-coloring g_f of V as follows: we let $g_f(v)$ be equal to the smallest i such that v is NOT a color of any vertex in W_i. Assume that g_f is not a proper coloring, i.e., that some e_j is monochromatic of some color i under g_f. But some $v' \in e_j$ must be $f(w_{i,j})$, and therefore $g_f(v') \neq i$, a contradiction. This proves that $Q(k,r) \leq k\ m(r,k)$.

Now, consider a complete k-partite k-uniform hypergraph $H = (W, A)$ with parts $W_1, \dots, W_k$ and $|W| < p(r,k)$. Let L be an arbitrary r-uniform list for W. Let $G = (V, E)$ be the hypergraph with $V = \bigcup_{w \in W} L(w)$ and $E = \{L(w) \mid w \in W\}$. Since $|E| = |W| < Q(r,k)$, there exists a k-coloring g of G. Define the coloring f_g of W as follows: if $w \in W_i$, choose the next in the cyclic order $1, 2, \dots, k$ after i number i' such that there is a vertex $v' \in L(w)$ with $g(v') = i'$ and let $f_g(w) = v'$. Since $L(w)$ is not monochromatic in g, we have $i' \neq i$. On the other hand, no v with $g(v) = i$ will be used to color a $w \in W_i$. Thus f_g is a proper coloring of H. This proves that $Q(k,r) \geq m(r,k)$.

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