Sharp Dirac's Theorem for DP-Critical Graphs

Anton Bernshteyn[*] Alexandr Kostochka[†]

Abstract

Correspondence coloring, or *DP-coloring*, is a generalization of list coloring introduced recently by Dvořák and Postle [9]. In this paper we establish a version of Dirac's theorem on the minimum number of edges in critical graphs [7] in the framework of DP-colorings. A corollary of our main result is a solution to the problem, posed by Kostochka and Stiebitz [13], of classifying list-critical graphs that satisfy Dirac's bound with equality.

1 Introduction

All graphs considered here are finite, undirected, and simple. We use $\mathbb{N}$ to denote the set of all nonnegative integers. For $k \in \mathbb{N}$, let $[k] \coloneqq \{1 \ldots, k\}$. For a set S, we use $\operatorname{Pow}(S)$ to denote the power set of S, i.e., the set of all subsets of S. For a function $f\colon A \to B$ and a subset $S \subseteq A$, we use $f|_S$ to denote the restriction of f to S. For a graph G, $V(G)$ and $E(G)$ denote the vertex and the edge sets of G, respectively. For a set $U \subseteq V(G)$, $G[U]$ is the subgraph of G induced by U. Let $G - U \coloneqq G[V(G) \setminus U]$, and for $u \in V(G)$, let $G - u \coloneqq G - \{u\}$. For two subsets U_1, $U_2 \subseteq V(G)$, $E_G(U_1, U_2) \subseteq E(G)$ denotes the set of all edges in G with one endpoint in U_1 and the other one in U_2. For $u \in V(G)$, $N_G(u) \subset V(G)$ denotes the set of all neighbors of u and $\deg_G(u) \coloneqq |N_G(u)|$ denotes the degree of u in G. For a subset $U \subseteq V(G)$, let $N_G(U) \coloneqq \bigcup_{u\in U} N_G(u)$ denote the neighborhood of U in G. A set $I \subseteq V(G)$ is *independent* if $I \cap N_G(I) = \emptyset$, i.e., if $uv \notin E(G)$ for all u, $v \in I$. We denote the family of all independent sets in a graph G by $\mathcal{I}(G)$. The complete graph on k vertices is denoted by K_k.

1.1 Critical graphs and theorems of Brooks, Dirac, and Gallai

Recall that a *proper coloring* of a graph G is a function $f\colon V(G) \to Y$, where Y is a set, whose elements are referred to as *colors*, such that $f(u) \neq f(v)$ for each edge $uv \in E(G)$.

[*]Department of Mathematics, University of Illinois at Urbana–Champaign, IL, USA, `bernsht2@illinois.edu`. Research of this author is supported by the Illinois Distinguished Fellowship.

[†]Department of Mathematics, University of Illinois at Urbana–Champaign, IL, USA and Sobolev Institute of Mathematics, Novosibirsk 630090, Russia, `kostochk@math.uiuc.edu`. Research of this author is supported in part by NSF grant DMS-1600592 and grants 15-01-05867 and 16-01-00499 of the Russian Foundation for Basic Research.

The least $k \in \mathbb{N}$ such that there exists a proper coloring $f : V(G) \to Y$ with $|Y| = k$ is called the *chromatic number* of G and is denoted by $\chi(G)$.

For $k \in \mathbb{N}$, a graph G is said to be $(k+1)$-*vertex-critical* if $\chi(G) = k+1$ but $\chi(G-u) \leq k$ for all $u \in V(G)$. We will only consider vertex-critical graphs in this paper, so for brevity we will call them simply *critical*. Since every graph G with $\chi(G) > k$ contains a $(k+1)$-critical subgraph, understanding the structure of critical graphs is crucial for the study of graph coloring. We will only consider $k \geq 3$, the case $k \leq 2$ being trivial (the only 1-critical graph is K_1, the only 2-critical graph is K_2, and the only 3-critical graphs are odd cycles).

Let $k \geq 3$ and suppose that G is a $(k+1)$-critical graph with n vertices and m edges. A classical problem in the study of critical graphs is to understand how small m can be depending on n and k. Evidently, $\delta(G) \geq k$, where $\delta(G)$ denotes the minimum degree of G; in particular, $2m \geq kn$. Brooks's Theorem is equivalent to the assertion that the only situation in which $2m = kn$ is when $G \cong K_{k+1}$:

Theorem 1.1 (Brooks [5, Theorem 14.4])**.** *Let $k \geq 3$ and let G be a $(k+1)$-critical graph distinct from K_{k+1}. Set $n := |V(G)|$ and $m := |E(G)|$. Then*

$$2m > kn.$$

Brooks's theorem was subsequently sharpened by Dirac, who established a linear in k lower bound on the difference $2m - kn$:

Theorem 1.2 (Dirac [7])**.** *Let $k \geq 3$ and let G be a $(k+1)$-critical graph distinct from K_{k+1}. Set $n := |V(G)|$ and $m := |E(G)|$. Then*

$$2m \geq kn + k - 2. \tag{1.1}$$

Bound (1.1) is sharp in the sense that for every $k \geq 3$, there exist $(k+1)$-critical graphs that satisfy $2m = kn + k - 2$. However, for each k, there are only finitely many such graphs; in fact, they admit a simple characterization, which we present below.

Definition 1.3. Let $k \geq 3$. A graph G is k-*Dirac* if its vertex set can be partitioned into three subsets V_1, V_2, V_3 such that (a) $|V_1| = k$, $|V_2| = k-1$, $|V_3| = 2$; (b) the graphs $G[V_1]$, $G[V_2]$ are complete; (c) each vertex in V_1 is adjacent to exactly one vertex in V_3, while each vertex in V_3 is adjacent to at least one vertex in V_1; (d) each vertex in V_2 is adjacent to every vertex in V_3; and (e) G has no other edges. We denote the family of all k-Dirac graphs by $\mathcal{D}_k$.

Theorem 1.4 (Dirac [8])**.** *Let $k \geq 3$ and let G be a $(k+1)$-critical graph distinct from K_{k+1}. Set $n := |V(G)|$ and $m := |E(G)|$. Then*

$$2m = kn + k - 2 \iff G \in \mathcal{D}_k.$$

As n goes to infinity, the gap between Dirac's lower bound and the sharp bound increases. In fact, Gallai [11] observed that the asymptotic density of large $(k+1)$-critical graphs distinct from K_{k+1} is strictly greater than $k/2$. However, Gallai's bound is stronger than (1.1) only for n at least quadratic in k.

1.2 List coloring

List coloring was introduced independently by Vizing [19] and Erdős, Rubin, and Taylor [10]. A *list assignment* for a graph G is a function $L \colon V(G) \to \mathrm{Pow}(Y)$, where Y is a set, whose elements, as in the case of ordinary colorings, are referred to as *colors*. For each $u \in V(G)$, the set $L(u)$ is called the *list* of u and its elements are said to be *available* for u. A proper coloring $f \colon V(G) \to Y$ is called an *L-coloring* if $f(u) \in L(u)$ for each $u \in V(G)$. A graph G is said to be *L-colorable* if it admits an L-coloring. The *list-chromatic number* $\chi_\ell(G)$ of G is the least $k \in \mathbb{N}$ such that G is L-colorable whenever L is a list assignment for G with $|L(u)| \geq k$ for all $u \in V(G)$. If $k \in \mathbb{N}$ and $L(u) = [k]$ for all $u \in V(G)$, then G is L-colorable if and only if it is k-colorable; in this sense, list coloring generalizes ordinary coloring. In particular, $\chi_\ell(G) \geq \chi(G)$ for all graphs G.

A list assignment L for a graph G is called a *degree list assignment* if $|L(u)| \geq \deg_G(u)$ for all $u \in V(G)$. A fundamental result of Borodin [6] and Erdős, Rubin, and Taylor [10], which can be seen as a generalization of Brooks's theorem to list colorings, provides a complete characterization of all graphs G that are not L-colorable with respect to some degree list assignment L.

Definition 1.5. A *Gallai tree* is a connected graph in which every block is either a clique or an odd cycle. A *Gallai forest* is a graph in which every connected component is a Gallai tree.

Theorem 1.6 (Borodin [6]; Erdős–Rubin–Taylor [10])**.** *Let G be a connected graph and let L be a degree list assignment for G. If G is not L-colorable, then G is a Gallai tree; furthermore, $|L(u)| = \deg_G(u)$ for all $u \in V(G)$ and if u, $v \in V(G)$ are two adjacent non-cut vertices, then $L(u) = L(v)$.*

Theorem 1.6 provides some useful information about the structure of critical graphs:

Corollary 1.7. *Let $k \geq 3$ and let G be a $(k+1)$-critical graph with minimum degree k. Set $D \coloneqq \{u \in V(G) : \deg_G(u) = k\}$. Then $G[D]$ is a Gallai forest.*

Corollary 1.7 was originally proved by Gallai [11] using a different method. It is crucial for the proof of Gallai's theorem on the asymptotic average degree of $(k+1)$-critical graphs.

The definition of critical graphs can be naturally extended to list colorings. A graph G is said to be *L-critical*, where L is a list assignment for G, if G is not L-colorable but for any $u \in V(G)$, the graph $G - u$ is $L|_{V(G)\setminus\{u\}}$-colorable. Note that if we set $L(u) \coloneqq [k]$ for all $u \in V(G)$, then G being L-critical is equivalent to it being $(k+1)$-critical. Repeating the argument used to prove Corollary 1.7, we obtain the following more general statement:

Corollary 1.8 (Kostochka–Stiebitz–Wirth [14])**.** *Let $k \geq 3$ and let G be a graph with minimum degree k. Suppose that L is a list assignment for G such that G is L-critical and $|L(u)| = k$ for all $u \in V(G)$. Set $D \coloneqq \{u \in V(G) : \deg_G(u) = k\}$. Then $G[D]$ is a Gallai forest.*

Corollary 1.7 can be used to prove a version of Gallai's theorem for list-critical graphs, i.e., to show that the average degree of a graph G distinct from K_{k+1} that is L-critical for some list assignment L with $|L(u)| = k$ for all $u \in V(G)$ has average degree strictly greater than $k/2$. On the other hand, list-critical graphs distinct from K_{k+1} do not, in general, admit a nontrivial lower bound on the difference $2m - kn$ that only depends on k (analogous to the one given by Dirac's Theorem 1.2 for $(k+1)$-critical graphs). Consider the following example, given in [13]. Fix $k \in \mathbb{N}$ and let G be the graph with vertex set $\{a_0, \ldots, a_k, b_0, \ldots, b_k\}$ of size $2(k+1)$ and edge set $\{a_i a_j, b_i b_j \,:\, i \neq j\} \cup \{a_0 b_0\}$. For each $i \in [k]$, let $L(a_i) = L(b_i) \coloneqq [k]$, and let $L(a_0) = L(b_0) \coloneqq \{0\} \cup [k-1]$. Then G is L-critical; however, $2|E(G)| - k|V(G)| = 2$.

Nonetheless, Theorem 1.2 can be extended to the list coloring framework if we restrict our attention to graphs that do not contain K_{k+1} as a *subgraph*:

Theorem 1.9 (Kostochka–Stiebitz [13])**.** *Let $k \geq 3$. Let G be a graph and let L be a list assignment for G such that G is L-critical and $|L(u)| = k$ for all $u \in V(G)$. Suppose that G does not contain a clique of size $k+1$. Set $n \coloneqq |V(G)|$ and $m \coloneqq |E(G)|$. Then*

$$2m \geq kn + k - 2.$$

Kostochka and Stiebitz [13] posed a problem of determining whether Theorem 1.4 holds for list critical graphs with no K_{k+1} as a subgraph as well. We show that the answer is positive; see Corollary 1.16.

1.3 DP-colorings and main results of this paper

In this paper we focus on a generalization of list coloring that was recently introduced by Dvořák and Postle [9]; they called it *correspondence coloring*, and we call it *DP-coloring* for short. Dvořák and Postle invented DP-coloring in order to approach an open problem about list coloring of planar graphs with no cycles of certain lengths.

Definition 1.10. Let G be a graph. A *cover* of G is a pair (L, H), where H is a graph and $L \colon V(G) \to \mathrm{Pow}(V(H))$ is a function, with the following properties:

- the sets $L(u)$, $u \in V(G)$, form a partition of $V(H)$;
- if u, $v \in V(G)$ and $L(v) \cap N_H(L(u)) \neq \emptyset$, then $v \in \{u\} \cup N_G(u)$;
- each of the graphs $H[L(u)]$, $u \in V(G)$, is complete;
- if $uv \in E(G)$, then $E_H(L(u), L(v))$ is a matching (not necessarily perfect and possibly empty).

Definition 1.11. Let G be a graph and let (L, H) be a cover of G. An (L, H)-*coloring* of G is an independent set $I \in \mathcal{I}(H)$ of size $|V(G)|$. Equivalently, $I \in \mathcal{I}(H)$ is an (L, H)-coloring of G if $I \cap L(u) \neq \emptyset$ for all $u \in V(G)$.

The *DP-chromatic number* $\chi_{DP}(G)$ of a graph G is the least $k \in \mathbb{N}$ such that G is (L, H)-colorable whenever (L, H) is a cover of G with $|L(u)| \geq k$ for all $u \in V(G)$.

In order to see that DP-colorings indeed generalize list colorings, consider a graph G and a list assignment L for G. Define a graph H as follows: Let

$$V(H) \coloneqq \{(u, c) \,:\, u \in V(G) \text{ and } c \in L(u)\},$$

and let

$$(u_1, c_1)(u_2, c_2) \in E(H) \ :\Longleftrightarrow\ (u_1 = u_2 \text{ and } c_1 \neq c_2) \text{ or } (u_1u_2 \in E(G) \text{ and } c_1 = c_2).$$

For $u \in V(G)$, set

$$\hat{L}(u) \coloneqq \{(u, c) \,:\, c \in L(u)\}.$$

Then $(\hat{L}, H)$ is a cover of G. Observe that there is a one-to-one correspondence between L-colorings and $(\hat{L}, H)$-colorings of G. Indeed, if f is an L-coloring of G, then the set

$$I_f \coloneqq \{(u, f(u)) \,:\, u \in V(G)\}$$

is an $(\hat{L}, H)$-coloring of G. Conversely, given an $(\hat{L}, H)$-coloring I of G, we can define an L-coloring f_I of G by the property $(u, f_I(u)) \in I$ for all $u \in V(G)$. This shows that list colorings can be identified with a subclass of DP-colorings. In particular, $\chi_{DP}(G) \geq \chi_\ell(G)$ for all graphs G.

Some upper bounds on list-chromatic numbers hold for DP-chromatic numbers as well. For instance, it is easy to see that $\chi_{DP}(G) \leq d+1$ for any d-degenerate graph G. Dvořák and Postle [9] pointed out that for any planar graph G, $\chi_{DP}(G) \leq 5$ and, moreover, $\chi_{DP}(G) \leq 3$, provided that the girth of G is at least 5 (these statements are extensions of classical results of Thomassen [17, 18] regarding list colorings). On the other hand, there are also some striking differences between DP- and list colorings. For example, even cycles are 2-list-colorable, while their DP-chromatic number is 3 (in particular, the orientation theorems of Alon–Tarsi [2] and the Bondy–Boppana–Siegel lemma (see [2]) do not extend to DP-colorings). Bernshteyn [3] showed that the DP-chromatic number of every graph with average degree d is $\Omega(d/\log d)$, i.e., almost linear in d (recall that due to a celebrated result of Alon [1], the list-chromatic number of such graphs is $\Omega(\log d)$, and this bound is best possible). On the other hand, Johansson's upper bound [12] on list chromatic numbers of triangle-free graphs also holds for DP-chromatic numbers [3].

A cover (L, H) of a graph G is a *degree cover* if $|L(u)| \geq \deg_G(u)$ for all $u \in V(G)$. Bernshteyn, Kostochka, and Pron [4] established the following generalization of Theorem 1.6:

Definition 1.12. A *GDP-tree* is a connected graph in which every block is either a clique or a cycle. A *GDP-forest* is a graph in which every connected component is a GDP-tree.

Theorem 1.13 ([4])**.** *Let G be a connected graph and let (L, H) be a degree cover of G. If G is not (L, H)-colorable, then G is a GDP-tree; furthermore, $|L(u)| = \deg_G(u)$ for all $u \in V(G)$ and if u, $v \in V(G)$ are two adjacent non-cut vertices, then $E_H(L(u), L(v))$ is a perfect matching.*

Let G be a graph and let (L, H) be a cover of G. We say that G is *(L, H)-critical* if G is not (L, H)-colorable but for every proper subset $U \subset V(G)$, there exists $I \in \mathcal{I}(H)$ such that $I \cap L(u) \neq \emptyset$ for all $u \in U$. Theorem 1.13 implies the following:

Corollary 1.14 ([4])**.** *Let $k \geq 3$ and let G be a graph with minimum degree k. Suppose that (L, H) is a cover of G such that G is (L, H)-critical and $|L(u)| = k$ for all $u \in V(G)$. Set $D \coloneqq \{u \in V(G) \,:\, \deg_G(u) = k\}$. Then $G[D]$ is a GDP-forest.*

Corollary 1.14 implies an extension of Gallai's theorem to DP-critical graphs.

The main result of this paper is a generalization of Theorem 1.9 to DP-critical graphs. In fact, we establish a sharp version that also generalizes Theorem 1.4:

Theorem 1.15. *Let $k \geq 3$. Let G be a graph and let (L, H) be a cover of G such that G is (L, H)-critical and $|L(u)| = k$ for all $u \in V(G)$. Suppose that G does not contain a clique of size $k+1$. Set $n \coloneqq |V(G)|$ and $m \coloneqq |E(G)|$. If $G \notin \mathcal{D}_k$, then*

$$2m > kn + k - 2.$$

An immediate corollary of Theorem 1.15 is the following version of Theorem 1.4 for list colorings:

Corollary 1.16. *Let $k \geq 3$. Let G be a graph and let L be a list assignment for G such that G is L-critical and $|L(u)| = k$ for all $u \in V(G)$. Suppose that G does not contain a clique of size $k+1$. Set $n \coloneqq |V(G)|$ and $m \coloneqq |E(G)|$. If $G \notin \mathcal{D}_k$, then*

$$2m > kn + k - 2.$$

Our proof of Theorem 1.15 is essentially inductive. As often is the case, having a stronger inductive assumption (due to considering DP-critical and not just list-critical graphs) allows for more flexibility in the proof. In particular, we do not know if our argument can be adapted to give a "DP-free" proof of Corollary 1.16.

2 Proof of Theorem 1.15: first observations

2.1 Set-up and notation

From now on, we fix a counterexample to Theorem 1.15; more precisely, we fix the following data:

- an integer $k \geq 3$;
- a graph G with n vertices and m edges such that $G \notin \mathcal{D}_k$, G does not contain a clique of size $k+1$, and
$$2m \leq kn + k - 2; \tag{2.1}$$
- a cover (L, H) of G such that $|L(u)| = k$ for all $u \in V(G)$ and G is (L, H)-critical.

Furthermore, we assume that G is a counterexample with the fewest vertices.

For brevity, we denote $V \coloneqq V(G)$ and $E \coloneqq E(G)$. For a subset $U \subseteq V$, set $U^c \coloneqq V \setminus U$ denote the complement of U in V. For $u \in V$ and $U \subseteq V$, set

$$\deg(u) \coloneqq \deg_G(u); \quad \deg_U(u) \coloneqq |U \cap N_G(u)|.$$

For $u \in V$, set

$$\varepsilon(u) \coloneqq \deg(u) - k,$$

and for $U \subseteq V$, define

$$\varepsilon(U) \coloneqq \sum_{u \in U} \varepsilon(u).$$

Note that inequality (2.1) is equivalent to

$$\varepsilon(V) \leq k - 2. \tag{2.2}$$

Since G is (L, H)-critical, we have $\varepsilon(u) \geq 0$ for all $u \in V$. Let

$$D \coloneqq \{u \in V \,:\, \deg(u) = k\} = \{u \in V \,:\, \varepsilon(u) = 0\}.$$

Since $\varepsilon(u) \geq 1$ for every $u \in D^c$, (2.2) yields $|D^c| \leq k - 2$. Since $n \geq k + 1$, D is nonempty, so Corollary 1.14 implies that $G[D]$ is a GDP-forest.

From now on, we refer to the vertices of H as *colors* and to the independent sets in H as *colorings*. For I, $I' \in \mathcal{I}(H)$, we say that I *extends* I' if $I \supseteq I'$. For $I \in \mathcal{I}(H)$, let

$$\mathrm{dom}(I) \coloneqq \{u \in V \,:\, I \cap L(u) \neq \emptyset\}.$$

Since G is (L, H)-critical, there is no coloring I with $\mathrm{dom}(I) = V$, but for every proper subset $U \subset V$, there exists a coloring I with $\mathrm{dom}(I) = U$.

For $I \in \mathcal{I}(H)$ and $u \in (\mathrm{dom}(I))^c$, let

$$L_I(u) \coloneqq L(u) \setminus N_H(I).$$

In other words, $L_I(u)$ is the set of all colors available for u in a coloring extending I.

For $u \in V$ and $U \subseteq V$, let

$$\varphi_U(u) \coloneqq \deg_U(u) - \varepsilon(u).$$

Note that

$$\varphi_U(u) = \deg_U(u) - (\deg(u) - k) = k - (\deg(u) - \deg_U(u)) = k - \deg_{U^c}(u).$$

In particular, if $I \in \mathcal{I}(H)$ is a coloring such that $\mathrm{dom}(I) = U^c$, then for all $u \in U$,

$$|L_I(u)| \geq \varphi_U(u). \tag{2.3}$$

Finally, for $u \in D$ and for any $U \subseteq V$, we have $\varphi_U(u) = \deg_U(u)$.

2.2 A property of GDP-forests

The following simple property of GDP-forests will be quite useful:

Proposition 2.1. *Let F be a GDP-forest of maximum degree at most k not containing a clique of size $k + 1$. Then*

$$\sum_{u \in V(F)} (k - \deg_F(u)) \geq k, \tag{2.4}$$

with equality only if $F \cong K_1$ or $F \cong K_k$.

Proof. It suffices to establish the proposition for the case when F is connected, i.e., a GDP-tree. If F is 2-connected, i.e., a clique or a cycle, then the statement follows via a simple calculation. It remains to notice that adding a leaf block to a GDP-tree of maximum degree at most k cannot decrease the quantity on the left-hand side of (2.4). ■

Corollary 2.2. *Suppose that $U \subseteq D$ is the vertex set of a connected component of $G[D]$. Then the number of edges in G between U and D^c is at least k, with equality only if $G[U] \cong K_k$.*

Proof. The number of edges between U and D^c is precisely

$$\sum_{u \in U} \deg_{U^c}(u) = \sum_{u \in U} (\deg(u) - \deg_U(u)) = \sum_{u \in U} (k - \deg_U(u)).$$

By Proposition 2.1, this quantity is at least k with equality only if $G[U] \cong K_1$ or $G[U] \cong K_k$. However, $G[U] \not\cong K_1$ since $|D^c| \leq k-2$ while $\deg(u) = k$ for all $u \in U$. ■

2.3 Enhanced vertices

The following definition will play a crucial role in our argument. Let U be a nonempty subset of V and let $I \in \mathcal{I}(H)$ be a coloring with $\mathrm{dom}(I) = U^c$. We say that a vertex $u \in U \cap D$ is *enhanced* by I (or I *enhances* u) if $|L_I(u)| > \deg_U(u)$. The importance of this notion stems from the following lemma:

Lemma 2.3. (i) *Let I, $I' \in \mathcal{I}(H)$ be colorings such that I' extends I and suppose that $u \in (\mathrm{dom}(I'))^c \cap D$ is a vertex enhanced by I. Then u is also enhanced by I'.*

(ii) *Let $U \subseteq D$ and let $I \in \mathcal{I}(H)$ be a coloring with $\mathrm{dom}(I) = U^c$. Let $U' \subseteq U$ be a subset such that the graph $G[U']$ is connected and suppose that U' contains a vertex enhanced by I. Then there exists a coloring $I' \in \mathcal{I}(H)$ with $\mathrm{dom}(I') = U^c \cup U'$ that extends I.*

(iii) *Let $U \subseteq V$ and let $I \in \mathcal{I}(H)$ be a coloring with $\mathrm{dom}(I) = U^c$. Suppose that I enhances at least one vertex in each connected component of $G[U \cap D]$. Then there does not exist a coloring $I' \in \mathcal{I}(H)$ with $\mathrm{dom}(I') \supseteq D^c$ that extends I.*

Proof. Since (i) is an immediate corollary of the definition and (ii) follows from Theorem 1.13, it only remains to prove (iii). To that end, suppose that I' is such a coloring. Without loss of generality, we may assume that $\mathrm{dom}(I') = U^c \cup D^c$. By (i), I' enhances at least one vertex in every connected component of $G[U \cap D]$. Applying (ii) to each connected component of $G[U \cap D]$, we can extend I' to an (L, H)-coloring of G; a contradiction. ■

The next lemma gives a convenient sufficient condition under which a given coloring can be extended so that the resulting coloring enhances a particular vertex:

Lemma 2.4. *Let $U \subseteq V$ and let $I \in \mathcal{I}(H)$ be a coloring with $\mathrm{dom}(I) = U^c$. Suppose that $u \in U \cap D$ and $A \subseteq U \cap N_G(u)$ is an independent set in G. Moreover, suppose that*

$$\min\{\varphi_U(v) \,:\, v \in A\} > 0 \quad \textit{and} \quad \sum_{v \in A} \varphi_U(v) > \deg_U(u).$$

Then there is a coloring $I' \in \mathcal{I}(H)$ with domain $U^c \cup A$ that extends I and enhances u.

Proof. Since the set A is independent and for all $v \in A$, $\varphi_U(v) > 0$ (and hence, by (2.3), $|L_I(v)| > 0$), any coloring I' with $\mathrm{dom}(I') \subseteq U^c \cup A$ can be extended to a coloring with domain $U^c \cup A$. Therefore, it suffices to find a coloring that extends I and enhances u and whose domain is contained in $U^c \cup A$.

If u is enhanced by I itself, then we are done, so assume that $|L_I(u)| = \deg_U(u)$. If for some $v \in A$, there is $x \in L_I(v)$ with no neighbor in $L_I(u)$, then u is enhanced by $I \cup \{x\}$, and we are done again. Thus, we may assume that for every $v \in A$, the matching $E_H(L_I(v), L_I(u))$ saturates $L_I(v)$. For each $v \in A$ and $x \in L_I(v)$, let $f(x)$ denote the neighbor of x in $L_I(u)$. Since $\sum_{v \in A} \varphi_U(v) > \deg_U(u)$, and hence, by (2.3), $\sum_{v\in A} |L_I(v)| > |L_I(u)|$, there exist distinct vertices $v, w \in A$ and colors $x \in L_I(v)$, $y \in L_I(w)$ such that $f(x) = f(y)$. Then u is enhanced by the coloring $I \cup \{x, y\}$, and the proof is complete. ∎

Corollary 2.5. *Suppose that u, u_1, $u_2 \in D$ are distinct vertices such that uu_1, $uu_2 \in E$, while $u_1u_2 \notin E$. Then the graph $G[D] - u_1 - u_2$ is disconnected.*

Proof. Note that
$$\varphi_V(u_1) = \varphi_V(u_2) = k \quad \text{and} \quad \deg(u) = k,$$
so, by Lemma 2.4, there exist $x_1 \in L(u_1)$ and $x_2 \in L(u_2)$ such that u is enhanced by the coloring $\{x_1, x_2\}$. Since for all $v \in D^c$,
$$|L_{\{x_1,x_2\}}(v)| \geq k - 2 \geq |D^c|,$$
we can extend $\{x_1, x_2\}$ to a coloring I with $\mathrm{dom}(I) = \{u_1, u_2\} \cup D^c$. Due to Lemma 2.3(iii), at least one connected component of the graph $G[D] - u_1 - u_2$ contains no vertices enhanced by I. Since, by Lemma 2.3(i), I enhances u, $G[D] - u_1 - u_2$ is disconnected, as desired. ∎

We will often apply Lemma 2.4 in the form of the following corollary:

Corollary 2.6. *Suppose that $u \in D$ and v_1, $v_2 \in D^c \cap N_G(u)$ are distinct vertices such that $v_1v_2 \notin E$. Let $U \subseteq D$ be any set containing u such that the graph $G[U]$ is connected. Then either $\min\{\varphi_U(v_1), \varphi_U(v_2)\} \leq 0$, or $\varphi_U(v_1) + \varphi_U(v_2) \leq \deg_U(u) + 2$.*

Proof. We only need to notice that
$$\varphi_{U\cup\{v_1,v_2\}}(v_i) = \varphi_U(v_i) \quad \text{for each } i \in \{1, 2\}, \quad \text{and} \quad \deg_{U\cup\{v_1,v_2\}}(u) = \deg_U(u) + 2. \quad ∎$$

The following observation can be viewed as an analog of Lemma 2.3(ii) for edges instead of vertices:

Lemma 2.7. *Let $U \subseteq D$ and let $I \in \mathcal{I}(H)$ be a coloring with $\mathrm{dom}(I) = U^c$. Let $U' \subseteq U$ be a subset such that the graph $G[U']$ is connected and let u_1, $u_2 \in U'$ be adjacent non-cut vertices in $G[U']$. Suppose that the matching $E_H(L_I(u_1), L_I(u_2))$ is not perfect. Then there exists a coloring $I' \in \mathcal{I}(H)$ with $\mathrm{dom}(I') = U^c \cup U'$ that extends I.*

Proof. Follows from Theorem 1.13. ∎

2.4 Vertices of small degree

In this subsection we establish some structural properties that G must possess if the minimum degree of the graph $G[D]$ is "small" (namely, at most 2).

Lemma 2.8. (i) *The minimum degree of $G[D]$ is at least 2.*

(ii) *If there is a vertex $u \in D$ such that $\deg_D(u) = 2$, then $|D^c| = k-2$, every vertex in D^c is adjacent to u, and $\varepsilon(v) = 1$ for all $v \in D^c$.*

(iii) *If the graph $G[D]$ has a connected component with at least 3 vertices of degree 2, then $G[D^c]$ is a disjoint union of cliques.*

(iv) *If the graph $G[D]$ has a connected component with at least 4 vertices of degree 2, then $G[D^c] \cong K_{k-2}$.*

Proof. (i) Note that a vertex $u \in D$ has precisely $k - \deg_D(u)$ neighbors in D^c. In particular, $k-2 \geq |D^c| \geq k - \deg_D(u)$, so $\deg_D(u) \geq 2$.

(ii) If $u \in D$ and $\deg_D(u) = 2$, then u has exactly $k-2$ neighbors in D^c, so $\varepsilon(D^c) = |D^c| = k-2$, which implies all the statements in (ii).

(iii) Let $U \subseteq D$ be the vertex set of a connected component of $G[D]$ such that $G[U]$ contains at least 3 vertices of degree 2 and suppose, towards a contradiction, that for some distinct vertices v_0, v_1, $v_2 \in D^c$, we have v_0v_1, $v_0v_2 \in E$, while $v_1v_2 \notin E$. By (ii), we have $|D^c| = k-2$, each vertex in D^c is adjacent to every vertex of degree 2 in $G[D]$, and $\varepsilon(v) = 1$ for all $v \in D^c$. Thus,

$$\varphi_{U\cup\{v_0,v_1,v_2\}}(v_i) = \deg_{U\cup\{v_0,v_1,v_2\}}(v_i) - \varepsilon(v_i) \geq 4-1 = 3 \quad \text{for each } i \in \{1,2\}.$$

Fix any vertex $u \in U$ such that $\deg_U(u) = 2$. Then

$$\deg_{U\cup\{v_0,v_1,v_2\}}(u) = \deg_U(u) + \deg_{\{v_0,v_1,v_2\}}(u) = 2+3 = 5.$$

Therefore, by Lemma 2.4, there exists a coloring $I \in \mathcal{I}(H)$ with domain $(U \cup \{v_0,v_1,v_2\})^c \cup \{v_1,v_2\} = (U \cup \{v_0\})^c$ that enhances u. But

$$\varphi_U(v_0) = \deg_U(v_0) - \varepsilon(v_0) \geq 3-1 = 2 > 0,$$

so I can be extended to $I' \in \mathcal{I}(H)$ with $\mathrm{dom}(I') = U^c$. This contradicts Lemma 2.3(iii).

(iv) If $U \subseteq D$ is the vertex set of a connected component of $G[D]$ with at least 4 vertices of degree 2 and v_1, $v_2 \in D^c$ are distinct nonadjacent vertices, then we have

$$\varphi_U(v_i) = \deg_U(v_i) - \varepsilon(v_i) \geq 4-1 = 3 \quad \text{for each } i \in \{1,2\},$$

so for every vertex $u \in U$ with $\deg_U(u) = 2$, we have

$$\varphi_U(v_1) + \varphi_U(v_2) \geq 3+3 > 4 = \deg_U(u) + 2;$$

a contradiction to Corollary 2.6. ∎

2.5 Terminal sets

We start this section by introducing some definitions and notation that will be used throughout the rest of the proof.

Definition 2.9. A *terminal set* is a subset $B \subseteq D$ such that $G[B]$ is a leaf block in a connected component of $G[D]$. For a terminal set B, $C_B \supseteq B$ denotes the vertex set of the connected component of $G[D]$ that contains B. A vertex $u \in D$ is *terminal* if it belongs to some terminal set B and is not a cut-vertex in $G[C_B]$.

By definition, a terminal set contains at most one non-terminal vertex. Since $G[D]$ is a GDP-forest, if B is a terminal set, then $G[B]$ is either a cycle or a clique.

Definition 2.10. A terminal set B is *dense* if $G[B]$ is not a cycle; otherwise, B is *sparse*.

Our proof hinges on the following key fact:

Lemma 2.11. *There exists a dense terminal set.*

Proof. Suppose that every terminal set is sparse. Since terminal vertices in sparse sets have degree 2 in $G[D]$, Lemma 3.8(ii) yields that $|D^c| = k-2$, each vertex in D^c is adjacent to every vertex of degree 2 in $G[D]$, and $\varepsilon(v) = 1$ for all $v \in D^c$. Furthermore, since every terminal set induces a cycle, each component of $G[D]$ contains at least 3 vertices of degree 2, and a component of $G[D]$ with exactly 3 vertices of degree 2 is isomorphic to a triangle. Therefore, by Lemma 3.8(iii, iv), $G[D^c]$ is a disjoint union of cliques and, unless every component of $G[D]$ is isomorphic to a triangle, $G[D^c] \cong K_{k-2}$.

Claim 2.11.1. $G[D^c] \not\cong K_{k-2}$.

Proof. Assume, towards a contradiction, that $G[D^c] \cong K_{k-2}$. Then every vertex in D^c has exactly $(k+1)-(k-3) = 4$ neighbors in D. Therefore, the number of vertices of degree 2 in $G[D]$ is at most 4. Since every component of $G[D]$ contains at least 3 vertices of degree 2, the graph $G[D]$ is connected. Since $|D| \geq 4$, $G[D]$ is not a triangle. Thus, it contains precisely 4 terminal vertices of degree 2; i.e., it either is a 4-cycle, or contains two leaf blocks, both of which are triangles.

CASE 1: $G[D]$ *is a* 4*-cycle.* We will show that in this case G is (L, H)-colorable. First, we make the following observation:

$$\text{Let } W_4 \text{ denote the 4-wheel. Then } \chi_{DP}(W_4) = 3. \tag{2.5}$$

Indeed, let (L', H') be a cover of W_4 with $|L'(u)| = 3$ for all $u \in V(W_4)$ and suppose that W_4 is not (L', H')-colorable. Let $v \in V(W_4)$ denote the center of W_4 and let $U := V(W_4) \setminus \{v\}$ (so $W_4[U]$ is a 4-cycle). Define a function $f \colon V(H') \to L'(v)$ by

$$f(x) = y \;:\Longleftrightarrow\; (x = y) \text{ or } (x \notin L'(v) \text{ and } xy \in E(H')).$$

Since $\deg_{W_4}(u) = 3$ for all $u \in U$, Theorem 1.13 implies that f is well-defined. Since W_4 is 3-colorable (in the sense of ordinary graph coloring), there exist an edge $u_1u_2 \in E(W_4)$ and a pair of colors $x_1 \in L'(u_1)$, $x_2 \in L'(u_2)$ such that $x_1x_2 \in E(H')$ and $f(x_1) \neq f(x_2)$. Note that

$u_1 \neq v$ since otherwise $f(x_1) = x_1 = f(x_2)$ by definition. Similarly, $u_2 \neq v$, so $\{u_1, u_2\} \subset U$. Let $y \coloneqq f(x_2)$. Then x_1 has no neighbor in $L'(u_2) \setminus N_{H'}(y)$, so $\{y\}$ can be extended to an (L', H')-coloring of W_4; a contradiction.

Let us now return to the graph G. Choose any vertex $v \in D^c$ and let $W \coloneqq G[\{v\} \cup D]$. Note that W is a 4-wheel. Fix an arbitrary coloring $I \in \mathcal{I}(H)$ with $\mathrm{dom}(I) = (\{v\} \cup D)^c$. For all $u \in \{v\} \cup D$, we have $|L_I(u)| \geq k - (k-3) = 3$, so by (2.5), I can be extended to an (L, H)-coloring of the entire graph G.

Case 2: *$G[D]$ contains two leaf blocks, both of which are triangles.* Since each vertex in D^c has only 4 neighbors in D, every non-terminal vertex in D has degree k in $G[D]$. Notice that every vertex of degree k in $G[D]$ is a cut-vertex. Indeed, if a vertex $u \in D$ is not a cut-vertex in $G[D]$, then the degree of any cut-vertex in the same block as u strictly exceeds the degree of u (since the blocks of the GDP-tree $G[D]$ are regular graphs). Thus, either the two terminal triangles share a cut-vertex (and, in particular, $k = 4$), or else, their cut-vertices are joined by an edge (and $k = 3$). The former option contradicts Corollary 2.5; the latter one implies $G \in \mathcal{D}_3$. ⊣

By Claim 2.11.1, $G[D^c]$ is a disjoint union of at least 2 cliques. In particular, every connected component of $G[D]$ is isomorphic to a triangle. Suppose that $G[D]$ has ℓ connected components (so $|D| = 3\ell$). If a vertex $v \in D^c$ belongs to a component of $G[D^c]$ of size r, then its degree in G is precisely $(r-1) + 3\ell$. On the other hand, $\deg(v) = k+1$. Thus, $k+1 = (r-1)+3\ell$, i.e., $r = k - 3\ell + 2$. In particular, $|D^c| = k-2$ is divisible by $k - 3\ell + 2$, so $\ell \geq 2$.

Case 1: *The set D^c is not independent, i.e., $k - 3\ell + 2 \geq 2$.* Let T_1, $T_2 \subset D$ (resp. C_1, $C_2 \subset D^c$) be the vertex sets of any two distinct connected components of $G[D]$ (resp. $G[D^c]$). For each $i \in \{1, 2\}$, fix a vertex $u_i \in T_i$ and a pair of distinct vertices v_{i1}, $v_{i2} \in C_i$. Set $U \coloneqq T_1 \cup T_2 \cup \{v_{11}, v_{12}, v_{21}, v_{22}\}$ and let $I \in \mathcal{I}(H)$ be such that $\mathrm{dom}(I) = U^c$. Note that

$$\varphi_U(v_{11}) = \varphi_U(v_{21}) = 7 - 1 = 6,$$

while

$$\deg_U(u_1) = 6,$$

so, by Lemma 2.4, there exist $x_{11} \in L_I(v_{11})$ and $x_{21} \in L_I(v_{21})$ such that $I' \coloneqq I \cup \{x_{11}, x_{21}\}$ is a coloring that enhances u_1. Now, upon setting $U' \coloneqq U \setminus \{v_{11}, v_{21}\}$, we obtain

$$\varphi_{U'}(v_{12}) = \varphi_{U'}(v_{22}) = 6 - 1 = 5,$$

while

$$\deg_{U'}(u_2) = 4,$$

so, by Lemma 2.4 again, we can choose $x_{12} \in L_{I'}(v_{12})$ and $x_{22} \in L_{I'}(v_{22})$ so that $I'' \coloneqq I' \cup \{x_{12}, x_{22}\}$ is a coloring that enhances both u_1 and u_2. However, the existence of such I'' contradicts Lemma 2.3(iii).

Case 2: *The set D^c is independent, i.e., $k - 3\ell + 2 = 1$.* In other words, we have $k = 3\ell - 1$. Since $\ell \geq 2$, we get $k \geq 6 - 1 = 5$, so $|D^c| = k - 2 \geq 3$. Let v_1, v_2, $v_3 \in D^c$ be any three distinct vertices in D^c and let $T \subset D$ be the vertex set of any connected component of $G[D]$. Fix a vertex $u \in T$, set $U := T \cup \{v_1, v_2, v_3\}$, and let $I \in \mathcal{I}(H)$ be such that $\mathrm{dom}(I) = U^c$. Note that

$$\varphi_U(v_1) = \varphi_U(v_2) = \varphi_U(v_3) = 3 - 1 = 2,$$

while

$$\deg_U(u) = 5.$$

Therefore, by Lemma 2.4, we can choose $x_1 \in L_I(v_1)$, $x_2 \in L_I(v_2)$, and $x_3 \in L_I(v_3)$ so that $I \cup \{x_1, x_2, x_3\}$ enhances u. This observation contradicts Lemma 2.3(iii) and finishes the proof. ∎

3 Dense terminal sets and their neighborhoods

3.1 Outline of the proof

Lemma 2.11 asserts that at least one terminal set is dense. In this section we explore the structural consequences of this assertion and eventually arrive at a contradiction.

Definition 3.1. A dense terminal set B is *heavy* if it is a largest dense terminal set contained in C_B.

By definition, if B is a dense terminal set, then C_B contains at least one heavy terminal set.

Definition 3.2. Let B be a terminal set. Let S_B denote the set of all vertices in B^c that are adjacent to every vertex in B and let $T_B := N_G(B) \setminus (B \cup S_B)$.

By definition, $S_B \subseteq D^c$; however, if $B \neq C_B$, then $T_B \cap D \neq \emptyset$.

The rest of the proof of Theorem 1.15 proceeds as follows. We start by showing that if B is a dense terminal set, then every vertex in T_B has "many" (namely at least $k-1$) neighbors outside of B (see Lemma 3.3). Intuitively, this should imply that the vertices in T_B can only have "very few" neighbors in B and thus "most" edges between B and D^c actually connect B to S_B. This intuition guides the proof of Corollary 3.6, which asserts that $G[B \cup S_B]$ is a clique of size k for every heavy terminal set B (however, the proof of Lemma 3.5, the main step towards Corollary 3.6, is somewhat lengthy and technical).

The fact that G is a *minimum* counterexample to Theorem 1.15 is only used once during the course of the proof, namely in establishing Lemma 3.10, which claims that for a heavy terminal set B, the graph $G[T_B]$ is a clique. The proof of Lemma 3.10 is also the only time when it is important to work in the more general setting of DP-colorings rather than just with list colorings. The proof proceeds by assuming, towards a contradiction, that there exist two nonadjacent vertices v_1, $v_2 \in T_B$ and letting G^* be the graph obtained from G by removing B and adding an edge between v_1 and v_2. Since G^* has fewer vertices than G, it cannot contain a counterexample to Theorem 1.15 as a subgraph. This fact can be used to eventually arrive at a contradiction. En route to that goal we investigate the properties of a certain cover of G^*—and that cover is not necessarily induced by a list assignment (even if (L, H) is).

With Lemma 3.10 at hand, we can pin down the structure of $G[S_B \cup T_B]$ very precisely, which is done in Lemmas 3.11 and 3.12 and in Corollary 3.13. The restrictiveness of these results precludes having "too many" dense terminal sets; this is made precise by Lemma 3.14, which asserts that at least one terminal set is sparse. However, due to Lemma 2.8, having a sparse terminal set leads to its own restrictions on the structure of $G[D^c]$, which finally yield a contradiction that finishes the proof of Theorem 1.15.

3.2 The set S_B is large

Lemma 3.3. *Let B be a dense terminal set. Suppose that $v \in T_B$. Then v has at least $k-1$ neighbors outside of B. If, moreover, there exist terminal vertices u_0, $u_1 \in B$ such that $u_0v \notin E$, $u_1v \in E$, then v has at least $k-1$ neighbors outside of C_B.*

Proof. Let u_0, $u_1 \in B$ be such that $u_0v \notin E$ and $u_1v \in E$. If one of u_0, u_1 is not terminal, then set $U := B$; otherwise, set $U := C_B$. Our goal is to show that v has at least $k-1$ neighbors outside of U.

Assume, towards a contradiction, that $\deg_{U^c}(v) \leq k-2$. Let $I \in \mathcal{I}(H)$ be such that $\mathrm{dom}(I) = (U \cup \{v\})^c$. By (2.3), we have $|L_I(v)| \geq \varphi_U(v) \geq k-(k-2) = 2$, so let x_1, x_2 be any two distinct elements of $L_I(v)$. Since $L_{I\cup\{x_1\}}(u_0) = L_{I\cup\{x_2\}}(u_0) = L_I(u_0)$, by Lemma 2.7, the matching $E_H(L_I(u_0), L_{I\cup\{x_i\}}(u_1))$ is perfect for each $i \in \{1,2\}$. This implies that the unique vertex in $L_I(u_1)$ that has no neighbor in $L_I(u_0)$ is adjacent to both x_1 and x_2, which is impossible. ■

Lemma 3.4. *Let B be a heavy terminal set. Then $|S_B| \geq k - |B|$.*

Proof. Let $S := S_B$ and let $T := T_B \cap D^c$. Set $b := |B|$, $s := |S|$, and $t := |T|$. Suppose, towards a contradiction, that $s \leq k-b-1$. Since each terminal vertex in B has exactly $k-(b-1)-s$ neighbors in T, the number of edges between B and T is at least $(b-1)(k-(b-1)-s)$. Also, by Lemma 3.3, each vertex in T has at least $k-1$ neighbors in B^c. Hence,

$$\begin{aligned}\varepsilon(V) &\geq \varepsilon(S) + \varepsilon(T)\\ &\geq s + (b-1)(k-(b-1)-s) + (k-1)t - kt\\ &= s + (b-1)(k-(b-1)-s) - t.\end{aligned}$$

Note that $s + t \leq |D^c| \leq k-2$, so $t \leq k-2-s$. Therefore,

$$s + (b-1)(k-(b-1)-s) - t \geq 2s + (b-1)(k-(b-1)-s) - k + 2.$$

Since $b \geq 4$, the last expression is decreasing in s, and hence

$$\begin{aligned}&2s + (b-1)(k-(b-1)-s) - k + 2\\ \geq\ &2(k-b-1) + (r-1)(k-(b-1)-(k-b-1)) - k + 2\\ =\ &k-2.\end{aligned}$$

On the other hand, $\varepsilon(V) \leq k-2$. This implies that none of the above inequalities are strict; in particular, the following statements hold:

1. $s = k - b - 1$ and $s + t = k - 2$, i.e., $t = b - 1$;

2. every vertex in T has exactly $k - 1$ neighbors in B^c;

3. $B \neq C_B$ and the cut vertex $u_0 \in B$ of $G[C_B]$ has no neighbors in T.

Since $s + t = k - 2$, we have $D^c = S \cup T$ and the degree of every vertex in D^c is exactly $k + 1$. Thus, every vertex in T has exactly $(k+1) - (k-1) = 2$ neighbors in $B \setminus \{u_0\}$. Notice that $|B \setminus \{u_0\}| = b - 1 \geq 3$, so, by Lemma 3.3, every vertex in T has $k - 1$ neighbors not in C_B. Therefore, there are no edges between T and $C_B \setminus B$.

Let u be any terminal vertex in $C_B \setminus B$. Since B is heavy, $\deg_D(u) \leq b - 1$, so $\deg_{D^c}(u) \geq k - b + 1$. But u cannot be adjacent to any vertices in T, so it can have at most $s = k - b - 1$ neighbors in D^c; a contradiction. ■

3.3 The graph $G[S_B]$

Lemma 3.5. *Let B be a heavy terminal set. Then $G[S_B]$ is a clique.*

Proof. Let $S \coloneqq S_B$ and suppose that $G[S]$ is not a clique, i.e., there exist distinct v_1, $v_2 \in S$ such that $v_1 v_2 \notin E$. Without loss of generality, we may assume that $\deg(v_1) \geq \deg(v_2)$.

We will proceed via a series of claims establishing a precise structure of $G[D^c]$, which will eventually lead to a contradiction. For the rest of the proof, we set $b \coloneqq |B|$ and $s \coloneqq |S|$.

Claim 3.5.1. (i) $|D^c| = k - b + 1$*;*
(ii) $\varepsilon(v_1) + \varepsilon(v_2) = b - 1$*; and*
(iii) *for every* $v \in D^c \setminus \{v_1, v_2\}$*,* $\varepsilon(v) = 1$*.*

Proof. Each terminal vertex $u \in B$ has exactly $k - b + 1$ neighbors in D^c; in particular, $|D^c| \geq k - b + 1$. By Corollary 2.6, we have

$$\begin{array}{rll} \text{either} & \min\{\varphi_B(v_1), \varphi_B(v_2)\} & \leq 0, \\ \text{or} & \varphi_B(v_1) + \varphi_B(v_2) & \leq b + 1. \end{array}$$

In the case when $\varphi_B(v_i) \leq 0$ for some $i \in \{1, 2\}$, we have $\varepsilon(v_i) = b - \varphi_B(v_i) \geq b$, so

$$\varepsilon(v_1) + \varepsilon(v_2) \geq b. \tag{3.1}$$

In the other case, i.e., when $\varphi_B(v_1) + \varphi_B(v_2) \leq b + 1$, we get

$$\varepsilon(v_1) + \varepsilon(v_2) = (b - \varphi_B(v_1)) + (b - \varphi_B(v_2)) \geq b - 1.$$

Hence

$$\varepsilon(D^c) \geq \varepsilon(v_1) + \varepsilon(v_2) + |D^c \setminus \{v_1, v_2\}| \geq (b - 1) + (k - b - 1) = k - 2. \tag{3.2}$$

Since $\varepsilon(D^c) \leq k - 2$, (3.1) does not hold and none of the inequalities in (3.2) are strict, which yields the claim. ⊣

Claim 3.5.2. *$D^c = S$ and $B = C_B$.*

Proof. Suppose, towards a contradiction, that there is a vertex $v \in D^c \setminus S$. Since, by Claim 3.5.1, $|D^c| = k - b + 1$, each terminal vertex in B is adjacent to every vertex in D^c. Therefore, $\deg_B(v) = b - 1$ and, due to Lemma 3.3, $\deg_{B^c}(v) \geq k - 1$. Then

$$\varepsilon(v) = \deg(v) - k \geq (b - 1) + (k - 1) - k = b - 2 > 1;$$

a contradiction to Claim 3.5.1.

Since $|S| = |D^c| = k - b + 1$, every vertex in B has $(b - 1) + (k - b + 1) = k$ neighbors in $B \cup S$, so there are no edges between B and $D \setminus B$; therefore, $B = C_B$. ⊣

Claim 3.5.3. *The graph $G[D]$ has no vertices of degree 2.*

Proof. Indeed, otherwise Lemma 2.8 would yield $|D^c| = k - 2$. Since $|D^c| = k - b + 1$, this implies $b = 3$; a contradiction. ⊣

Claim 3.5.4. *$s \geq 3$, i.e., $b \leq k - 2$.*

Proof. Suppose, towards a contradiction, that $s = 2$. We will argue that in this case $G \in \mathcal{D}_k$. Since, by Claim 3.5.1, $s = k - b + 1$, we have $b = k - 1$. In particular, since $b \geq 4$, we have $k \geq 5$. By Claim 3.5.1 again, $\varepsilon(S) = k - 2$, so there are exactly $(k - 2) + 2k - 2(k - 1) = k$ edges between S and $D \setminus B$. Let U be any connected component of $G[D]$ distinct from B. By Corollary 2.2, the number of edges between U and D^c is at least k, with equality only if $G[U] \cong K_k$; therefore, $D \setminus B = U$. Furthermore, every vertex in U has exactly one neighbor in S and each vertex in S has at least two neighbors in U (for its degree is at least $k + 1$), so $G \in \mathcal{D}_k$, as desired. ⊣

Claim 3.5.5. *$G[S \setminus \{v_1\}]$ is a clique.*

Proof. Suppose that for some distinct w_1, $w_2 \in S \setminus \{v_1\}$, we have $w_1w_2 \notin E$. Applying Claim 3.5.1 to w_1 and w_2 instead of v_1 and v_2, we obtain that $\varepsilon(v_1) = 1$. Since $\deg(v_1) \geq \deg(v_2)$, and thus $\varepsilon(v_1) \geq \varepsilon(v_2)$, we get $\varepsilon(v_2) = 1$ as well. But then $2 = \varepsilon(v_1) + \varepsilon(v_2) = b - 1$, i.e., $b = 3$; a contradiction. ⊣

Claim 3.5.6. $\deg_S(v_1) = 0$.

Proof. Suppose that $v \in S \setminus \{v_1, v_2\}$ is adjacent to v_1. Note that by Claim 3.5.5, v is also adjacent to v_2. Let $U \coloneqq B \cup \{v_1, v_2, v\}$ and let u be any vertex in B. Note that

$$\deg_U(u) = (b - 1) + 3 = b + 2.$$

On the other hand, since $\varepsilon(v_1) + \varepsilon(v_2) = b - 1$, for each $i \in \{1, 2\}$, we have $\varepsilon(v_i) \leq b - 2$, so

$$\varphi_U(v_i) = (b + 1) - \varepsilon(v_i) \geq (b + 1) - (b - 2) = 3 > 0;$$

moreover,

$$\varphi_U(v_1) + \varphi_U(v_2) = 2(b + 1) - (b - 1) = b + 3 > b + 2.$$

Therefore, by Lemma 2.4, for any $I \in \mathcal{I}(H)$ with $\mathrm{dom}(I) = U^c$, we can find $x_1 \in L_I(v_1)$ and $x_2 \in L_I(v_2)$ such that u is enhanced by $I' \coloneqq I \cup \{x_1, x_2\}$. Note that

$$|L_{I'}(v)| \geq \varphi_B(v) = b - 1 > 0,$$

so I' can be extended to a coloring with domain B^c, which contradicts Lemma 2.3. ⊣

Note that Claims 3.5.1, 3.5.4, and 3.5.6 imply that $\varepsilon(v_1) = b - 2$ and $\varepsilon(v) = 1$ for all $v \in S \setminus \{v_1\}$.

Claim 3.5.7. *Every terminal set distinct from B induces a clique of size k.*

Proof. Suppose that B' is a terminal set distinct from B and $b' := |B'| \leq k-1$. By Claim 3.5.3, $b' \geq 4$ and $G[B']$ is a clique. Thus, by Lemma 3.4, S contains at least $k - b'$ vertices that are adjacent to every vertex in B'. For all $v \in S \setminus \{v_1\}$, we have $\deg_{D\setminus B}(v) = 2$, so v cannot be adjacent to all the vertices in B'. Therefore, $|B'| = k - 1$ and v_1 is the only vertex in S adjacent to all the vertices in B'. But $\deg_{D\setminus B}(v_1) = (b-2) + k - b = k - 2 < k - 1$; a contradiction. $\dashv$

Claim 3.5.8. *There are exactly two terminal sets distinct from B.*

Proof. Suppose $D\setminus B$ contains ℓ terminal sets. By Claim 3.5.7, the number of edges between S and the terminal vertices of any terminal set B' distinct from B is at least $k-1$ and at most k. On the other hand, the number of edges between S and $D \setminus B$ is exactly $(k-2)+2(k-b) = 3k-2b-2$. Therefore, $\ell(k-1) \leq 3k-2b-2 \leq \ell k$, so $1 \leq \ell \leq 2$. However, if B' is a unique terminal set in $D \setminus B$, then $B' = D \setminus B$, and we have $3k - 2b - 2 = k$, i.e., $b = k - 1$, which contradicts Claim 3.5.4. Thus, $\ell = 2$, as desired. $\dashv$

Now we are ready to finish the argument. Let B_1 and B_2 denote the two terminal sets in $D \setminus B$. Recall that $D \setminus B = C_{B_1} \cup C_{B_2}$. Notice that v_1 is adjacent to at least one terminal vertex in $B_1 \cup B_2$; indeed, there are at least $2(k-1)$ edges between S and the terminal vertices in $B_1 \cup B_2$, while each vertex in $S \setminus \{v_1\}$ has 2 neighbors in $D \setminus B$, providing in total only $2(k-b)$ edges.

Without loss of generality, assume that v_1 is adjacent to at least one terminal vertex in B_1. Since v_1 has only $k-2$ neighbors in $D \setminus B$, Lemma 3.3 yields that v_1 has at least $k-1$ neighbors outside of C_{B_1}. Since v_1 has only $b \leq k-2$ neighbors outside of $C_{B_1} \cup C_{B_2}$, we see that $C_{B_1} \neq C_{B_2}$. Thus, $B_1 = C_{B_1}$ and $B_2 = C_{B_2}$. Therefore, v_1 is also adjacent to at least one terminal vertex in B_2 and, hence, has at least $k-1$ neighbors outside of B_2.

Notice that $2k = |E_G(B_1 \cup B_2, S)| = 3k - 2b - 2$, i.e., $k = 2b + 2$. Let $d_i := \deg_{B_i}(v_1)$. Then for each $i \in \{1, 2\}$, $d_i \geq k - 1 - b$. Since

$$b + d_1 + d_2 = \deg(v_1) = k + b - 2,$$

we obtain that $k + b - 2 \geq b + 2(k-1-b)$, i.e., $2b \geq k$, contradicting $k = 2b + 2$. ■

Corollary 3.6. *Let B be a heavy terminal set. Then $G[B \cup S_B]$ is a clique of size k.*

Proof. By Lemma 3.4, $|B \cup S_B| \geq k$; on the other hand, by Lemma 3.5, $G[B \cup S_B]$ is a clique, so $|B \cup S_B| \leq k$. ■

Corollary 3.7. *There does not exist a subset $U \subseteq V$ of size $k+1$ such that $G[U]$ is a complete graph minus an edge with the two nonadjacent vertices in D^c.*

Proof. Suppose, towards a contradiction, that U is such a set and let v_1, $v_2 \in U \cap D^c$ be the two nonadjacent vertices in U. Set $B \coloneqq U \cap D$. Note that $|B| \geq |U| - |D^c| \geq (k+1) - (k-2) = 3$.

Since for each $u \in B$, $\deg_U(u) = k$, there are no edges between B and U^c. In particular, $B = C_B$. If $|B| \geq 4$, then B is heavy and $U = B \cup S_B$, which is impossible due to Corollary 3.6. Therefore, $|B| = 3$. Thus, $|U \setminus B| = (k+1) - 3 = k - 2$, so $D^c = U \setminus B$. By Lemma 3.8(iii), $G[D^c]$ is a disjoint union of cliques. On the other hand, $G[D^c]$ is a complete graph minus the edge $v_1 v_2$. The only possibility then is that $|D^c| = 2$, i.e., $k = 4$. But then $G \in \mathcal{D}_4$. ∎

3.4 The graph $G[T_B]$

In this section we show that if B is a heavy terminal set, then $G[T_B]$ is a clique. However, in order for some of our arguments to go through, we need to establish some of the results for the more general case when B is any terminal set such that $G[B \cup S_B]$ is a clique of size k (which holds for heavy sets due to Corollary 3.6).

Lemma 3.8. *Let B be a terminal set such that $G[B \cup S_B]$ is a clique of size k. Then every vertex in S_B has at most $|B| - 1$ neighbors outside of $B \cup S_B$.*

Proof. Set $S \coloneqq S_B$. Let $v \in S$ and suppose that v has d neighbors outside of $B \cup S$. Then

$$\varepsilon(v) = \deg_{B \cup S}(v) + \deg_{(B \cup S)^c}(v) - k = (k-1) + d - k = d - 1,$$

so, using that $|S| = k - |B|$, we obtain

$$k - 2 \geq \varepsilon(V) = \varepsilon(S) + \varepsilon(D^c \setminus S) \geq (d-1) + (k - |B| - 1) + |D^c \setminus S|,$$

i.e., $d \leq |B| - |D^c \setminus S|$. It remains to notice that $D^c \setminus S \neq \emptyset$, since each terminal vertex in B has a neighbor in $D^c \setminus S$. ∎

Lemma 3.9. *Let B be a terminal set such that $G[B \cup S_B]$ is a clique of size k. Let $I \in \mathcal{I}(H)$ be such that $\mathrm{dom}(I) = (B \cup S_B)^c$. Then for any two distinct vertices u_1, $u_2 \in B$, the matching $E_H(L_I(u_1), L_I(u_2))$ is perfect.*

Proof. Set $S \coloneqq S_B$. Note that $|L_I(u)| = k - 1$ for all $u \in B$. Moreover, by Lemma 3.8, $|L_I(v)| \geq k - (|B| - 1) = k - |B| + 1$ for all $v \in S$. Let u_1, u_2 be two distinct vertices in B. Suppose, towards a contradiction, that $x \in L_I(u_1)$ has no neighbor in $L_I(u_2)$. For each $v \in S$, let $L'(v) \coloneqq L_I(v) \setminus N_H(x)$. Then $|L'(v)| \geq k - |B| = |S|$ for all $v \in S$, so there is a coloring $I' \in \mathcal{I}(H)$ with $\mathrm{dom}(I') = S$ such that $I' \subseteq \bigcup_{v \in S} L'(v)$. Note that $I \cup I'$ is a coloring with domain B^c; moreover, $x \in L_{I \cup I'}(u_1)$, which implies that the matching $E_H(L_{I \cup I'}(u_1), L_{I \cup I'}(u_2))$ is not perfect. Due to Lemma 2.7, $I \cup I'$ can be extended to an (L, H)-coloring of G; a contradiction. ∎

Lemma 3.10. *Let B be a terminal set such that $G[B \cup S_B]$ is a clique of size k. Then $G[T_B]$ is a clique of size at least 2.*

Proof. Set $S \coloneqq S_B$ and $T \coloneqq T_B$. First, observe that $|T| \geq 2$: Since $G[B \cup S]$ is a clique of size k, each vertex in B has a (unique) neighbor in T; thus, if $|T| = 1$, then the only vertex in T has to be adjacent to all the vertices in B, which contradicts the way T is defined.

Now suppose, towards a contradiction, that v_1, $v_2 \in T$ are two distinct nonadjacent vertices. For each $i \in \{1,2\}$, choose a neighbor $u_i \in B$ of v_i. Since every vertex in B has only one neighbor outside of $B \cup S$, u_1v_2, $u_2v_1 \notin E$. Note that, by Lemma 3.9, there are at least $k-1$ edges between $L(u_1)$ and $L(u_2)$. Let H' be the graph obtained from H by adding, if necessary, a single edge between $L(u_1)$ and $L(u_2)$ that completes a perfect matching between those two sets. Let H^* be the graph obtained from H by adding a matching M between $L(v_1)$ and $L(v_2)$ in which $x_1 \in L(v_1)$ is adjacent to $x_2 \in L(v_2)$ if and only if there exist $y_1 \in L(u_1)$, $y_2 \in L(u_2)$ such that $x_1y_1y_2x_2$ is a path in H'. Observe that (L, H^*) is a cover of the graph G^* obtained from G by adding the edge v_1v_2.

Claim 3.10.1. *There is no independent set* $I \in \mathcal{I}(H^*)$ *with* $\mathrm{dom}(I) = (B \cup S)^c$.

Proof. Assume, towards a contradiction, that $I \in \mathcal{I}(H^*)$ is such that $\mathrm{dom}(I) = (B \cup S)^c$. Since, in particular, $I \in \mathcal{I}(H)$, Lemma 3.9 guarantees that the edges of H between $L_I(u_1)$ and $L_I(u_2)$ form a perfect matching of size $k-1$. For each $i \in \{1,2\}$, let y_i be the unique element of $L(u_i) \setminus L_I(u_i)$. Then y_1y_2 is an edge in H'. However, since $y_i \notin L_I(u_i)$, the unique element of $I \cap L(v_i)$, which we denote by x_i, is adjacent to y_i in H. Therefore, $x_1y_1y_2x_2$ is a path in H', so x_1x_2 is an edge in H^*; a contradiction. $\dashv$

Let $W \subseteq (B \cup S)^c$ be an inclusion-minimal subset for which there is no $I \in \mathcal{I}(H^*)$ with $\mathrm{dom}(I) = W$. Since G is (L,H)-critical, $G^*[W]$ is not a subgraph of G, so $\{v_1, v_2\} \subseteq W$. Since for all $v \in W$, $\deg(v) \geq \deg_{G^*[W]}(v)$, we have

$$\varepsilon(W) \geq \sum_{v \in W} (\deg_{G^*[W]}(v) - k).$$

In particular,

$$\sum_{v \in W} (\deg_{G^*[W]}(v) - k) \leq k - 2.$$

Due to the minimality of G, either $G^*[W] \in \mathcal{D}_k$, or else, $G^*[W]$ contains a clique of size $k+1$.

If $G^*[W] \in \mathcal{D}_k$, then

$$\sum_{v \in W} (\deg_{G^*}(v) - k) = k - 2.$$

Therefore, $\deg(v) = \deg_{G^*[W]}(v)$ for all $v \in W$ and $D^c \subseteq W$. The second condition implies that $S = \emptyset$, so $|B| = k$. The first condition then shows that $T = \{v_1, v_2\}$ and, moreover, the only neighbors of v_1 and v_2 in B are u_1 and u_2. In other words, $|B| \leq 2$, so $k \leq 2$, which is impossible.

Thus, $G^*[W]$ contains a clique of size $k+1$. Since G does not contain such a clique, there exists a set $U \subseteq (B \cup S \cup \{v_1, v_2\})^c$ of size $k-1$ such that the graph $G[U \cup \{v_1, v_2\}]$ is isomorphic to K_{k+1} minus the edge v_1v_2. Note that U is not a subset of D^c, since $|D^c| < k-1$, so the set $B' := U \cap D$ is nonempty. Let $S' := U \setminus B'$. Notice that each vertex in B' has k neighbors in $U \cup \{v_1, v_2\}$, so there are no edges between B' and $(U \cup \{v_1, v_2\})^c$. Due to Corollary 3.7, $\{v_1, v_2\} \nsubseteq D^c$, so we can assume, without loss of generality, that $v_2 \in D$ and let $B^* := B' \cup \{v_2\}$. Then $G[B \cup B^*]$ is a connected component of $G[D]$, with u_2v_2 being a unique edge between terminal sets B and B^*. Note that $S' = S_{B^*}$ and $G[B^* \cup S']$ is a clique of size k. Moreover, $v_1u_2 \notin E$ and $\{v_1, u_2\} \in T_{B^*}$. Thus, we can apply the above reasoning to B^* in place of B and v_1, u_2 in place of v_1, v_2. As a result, we see that $G[B \cup S \cup \{v_1\}]$ is isomorphic to K_{k+1} minus the edge v_1u_2. Therefore,

$$\varepsilon(v_1) \geq \deg_B(v_1) + \deg_{B^*}(v_1) - k = (k-1) + (k-1) - k = k - 2.$$

Thus, $D^c = \{v_1\}$, $S = S' = \emptyset$, and $|B| = |B^*| = k$. This implies that $G \in \mathcal{D}_k$. ■

3.5 The graph $G[S_B \cup T_B]$

Lemma 3.11. *Let B be a heavy terminal set. Then:*
(i) $|T_B| = 2$;
(ii) $D^c \subseteq S_B \cup T_B$;
(iii) *each vertex in T_B has exactly $k-1$ neighbors outside of B; and*
(iv) $\varepsilon(v) = 1$ *for all* $v \in S_B$.

Proof. Let $S \coloneqq S_B$ and $T \coloneqq T_B$. By Corollary 3.6, $G[S \cup B]$ is a clique of size k, so, by Lemma 3.10, $G[T]$ is a clique of size at least 2.

Suppose that (i) does not hold, i.e., $|T| \geq 3$. If T contains at most one vertex with only $k-1$ neighbors outside of B, then

$$\varepsilon(S) + \varepsilon(T) \geq (k - |B|) + (|B| + k|T| - 1) - k|T| = k - 1;$$

a contradiction. Thus, there exist two distinct vertices v_1, $v_2 \in T$ such that $\deg_{B^c}(v_1) = \deg_{B^c}(v_2) = k - 1$. Since $|T| \geq 3$ and every vertex in B has exactly one neighbor in T, there exists a vertex $u_0 \in B$ such that u_0v_1, $u_0v_2 \notin E$. Also, we can choose a vertex $u_1 \in B$ with $u_1v_1 \in E$; note that $u_1v_2 \notin E$. Let $I \in \mathcal{I}(H)$ be such that $\mathrm{dom}(I) = (B \cup \{v_1, v_2\})^c$. Then

$$\varphi_{B\cup\{v_1,v_2\}}(v_1) = \varphi_{B\cup\{v_1,v_2\}}(v_2) = k - (k-2) = 2.$$

(Here we are using the fact that v_1 and v_2 are adjacent to each other.) Let x_1, x_2 be any two distinct elements of $L_I(v_1)$ and choose y_1, $y_2 \in L_I(v_2)$ so that x_1y_1, $x_2y_2 \notin E(H)$. Since $L_{I\cup\{x_1,y_1\}}(u_0) = L_{I\cup\{x_2,y_2\}}(u_0) = L_I(u_0)$ and for each $i \in \{1,2\}$, $L_{I\cup\{x_i,y_i\}}(u_1) = L_{I\cup\{x_i\}}(u_1)$, Lemma 2.7 yields that for each $i \in \{1, 2\}$, the matching $E_H(L_I(u_0), L_{I\cup\{x_i\}}(u_1))$ is perfect. But this implies that $L_{I\cup\{x_1\}}(u_1) = L_{I\cup\{x_2\}}(u_1)$. This contradiction proves (i).

In view of (i), we now have

$$\varepsilon(D^c) \geq \varepsilon(S) + \varepsilon(T) \geq (k - |B|) + (|B| + 2(k-1)) - 2k = k - 2, \tag{3.3}$$

so none of the inequalities in (3.3) can be strict. This yields (ii), (iii), and (iv). ■

Lemma 3.12. *Let B be a heavy terminal set. Then $B = C_B$.*

Proof. Suppose, towards a contradiction, that $B \neq C_B$ and let B' be any other terminal set with $C_{B'} = C_B$.

Claim 3.12.1. *B' is heavy.*

Proof. Since $B \neq C_B$, we have $T_B \not\subseteq D^c$, so, by Lemma 3.11(ii), we have $|D^c| = (k - |B|) + 2 - 1 = k - |B| + 1$. On the other hand, each terminal vertex in B' has at least $k - |B'| + 1$ neighbors in D^c. Therefore, $G[B']$ is a clique and $|B'| \geq |B|$. Since B is heavy, B' is also heavy, as desired. ⊣

Note that $S_{B'} \subseteq D^c \subseteq S_B \cup T_B$; since every vertex in S_B has only 2 neighbors in $(B \cup S_B)^c$, we conclude that $S_{B'} \subseteq T_B$. Moreover, one of the 2 vertices in T_B belongs to C_B, so $|S_{B'}| \leq 1$ and hence $|B| = |B'| \geq k-1$. The unique vertex in $T_B \cap D^c$ is adjacent to all the terminal vertices in B, of which there are $|B|-1$. Since the set of all terminal vertices in B is disjoint from $B' \cup S_{B'} \cup T_{B'}$, we get that $|B| - 1 \leq \max\{2, k-2\}$, i.e., $|B| \leq \max\{3, k-1\}$. Since $|B| \geq 4$, we see that $|B| = |B'| = k-1$. But then the only vertex in $T_B \cap D^c$ belongs to $S_{B'}$, so it is adjacent to at most 2 terminal vertices in B, which again yields $|B| \leq 3$; a contradiction. ∎

Corollary 3.13. *Let B be a heavy terminal set. Then $D^c = S_B \cup T_B$.*

Proof. Follows immediately by Lemma 3.12 and Lemma 3.11(ii). ∎

3.6 Finishing the proof of Theorem 1.15

Lemma 3.14. *There exists a sparse terminal set.*

Proof. Suppose, towards a contradiction, that every terminal set is dense. Since in that case each connected component of $G[D]$ contains a heavy set, Lemma 3.12 implies that every connected component of $G[D]$ is a clique of size at least 4. Moreover, due to Corollary 3.13, the size of every connected component of $G[D]$ is precisely $k - |D^c| + 2 =: b$. Note that due to Lemma 3.11(iii), the graph $G[D]$ is disconnected.

Let B_1 and B_2 be the vertex sets of any two distinct connected components of $G[D]$. Lemma 3.11(iv) implies that $S_{B_1} \cap S_{B_2} = \emptyset$. Since $D^c = S_{B_1} \cup T_{B_1} = S_{B_2} \cup T_{B_2}$, it follows that $S_{B_1} \subseteq T_{B_2}$ and $S_{B_2} \subseteq T_{B_1}$. Therefore, $k - b \leq 2$, i.e., $b \in \{k-2, k-1, k\}$. Now it remains to check the three remaining cases.

Case 1: $b = k-2$. Let B be the vertex set of any connected component of $G[D]$. Set $T_B =: \{v_1, v_2\}$ and let u_1, $u_2 \in B$ be such that u_1v_1, $u_2v_2 \in E$. Choose any $x \in L(v_1)$. Note that $|L_{\{x\}}(v_2)| \geq k-1 \geq 2$, so we can choose $y \in L_{\{x\}}(v_2)$ in such a way that $E_H(L_{\{x,y\}}(u_1), L_{\{x,y\}}(u_2))$ is not a perfect matching. For all $u \in D \setminus B$, we have $|L_{\{x,y\}}(u)| \geq k-2$ and the size of every connected component of $G[D \setminus B]$ is $k-2$. Therefore, there exists a coloring $I \in \mathcal{I}(H)$ with $\mathrm{dom}(I) = D \setminus B$ such that $I \cup \{x, y\} \in \mathcal{I}(H)$. But $\mathrm{dom}(I \cup \{x, y\}) = (B \cup S_B)^c$ and the matching $E_H(L_{I\cup\{x,y\}}(u_1), L_{I\cup\{x,y\}}(u_1))$ is not perfect—a contradiction to Lemma 3.9.

Case 2: $b = k-1$. Let B_1 and B_2 be the vertex sets of any two distinct connected components of $G[D]$. There is a vertex in D^c adjacent to all the $k-1$ vertices in B_1. However, every vertex in D^c has at most $\max\{2, k-2\} < k-1$ neighbors in B_2^c; a contradiction.

Case 3: $b = k$. In this case, $G[D^c] \cong K_2$ and there are exactly k edges between D^c and every connected component of $G[D]$. On the other hand, if B is the vertex set of a connected component of $G[D]$, then there are exactly $2(k-2) < 2k$ edges between D^c and $D \setminus B$. Thus, the graph $G[D \setminus B]$ is connected. Moreover, $k \geq 4$, for $2 \cdot (3-2) = 2 < 3$. Let $B' := D \setminus B$ (so $G[B']$ is a clique of size k). Set $D^c =: \{v_1, v_2\}$ an let u_1, $u_2 \in B$, u_1', $u_2' \in B'$ be such that u_1v_1, u_2v_2, $u_1'v_1$, $u_2'v_2 \in E$. Choose any $x \in L(v_1)$. Note that $|L_{\{x\}}(v_2)| \geq k-1 \geq 3$. There is at most one element $y \in L_{\{x\}}(v_2)$ such that $E_H(L_{\{x,y\}}(u_1), L_{\{x,y\}}(u_2))$ is a perfect matching; similarly for u_1' and u_2'. Therefore, there exists $z \in L_{\{x\}}(v_2)$ such that neither $E_H(L_{\{x,z\}}(u_1), L_{\{x,z\}}(u_2))$, nor $E_H(L_{\{x,z\}}(u_1'), L_{\{x,z\}}(u_2'))$ are perfect matchings. Thus, $\{x, z\}$ can be extended to an (L, H)-coloring of G; a contradiction. ∎

Now we are ready to finish the proof of Theorem 1.15. Let B be a heavy terminal set (which exists by Lemma 2.11) and let B' be a sparse terminal set (which exists by Lemma 3.14). Note that by Lemma 3.12, $B = C_B$ and every terminal set in $C_{B'}$ is sparse. In particular, $G[C_{B'}]$ contains at least 3 vertices of degree 2. Thus, every vertex in D^c has at least 3 neighbors in $C_{B'} \subseteq (B \cup S_B)^c$. On the other hand, by Lemma 3.11(iv), a vertex in S_B can have at most 2 neighbors in $(B \cup S_B)^c$. Therefore, $S_B = \emptyset$. Due to Corollary 3.13, we obtain $D^c = T_B$, i.e., $|D^c| = 2$. On the other hand, $|D^c| = k - 2$, so $k = 4$. But each vertex in T_B has at least 4 neighbors outside of B (1 in T_B and 3 in $C_{B'}$), which contradicts Lemma 3.11(iii).

4 Concluding remarks

In [4], the notion of DP-coloring was naturally extended to multigraphs (with no loops). The only difference from the graph case is that if distinct vertices $u, v \in V(G)$ are connected by t edges in G, then the set $E_H(L(u), L(v))$ is a union of t matchings (not necessarily perfect and possibly empty). Bounding the difference $2|E(G)| - k|V(G)|$ for DP-critical multigraphs G appears to be a challenging problem.

Definition 4.1. For $k \geq 3$, a *k-brick* is a k-regular multigraph whose underlying simple graph is either a clique or cycle and in which the multiplicities of all edges are the same.

Note that for a k-brick G, $2|E(G)| = k|V(G)|$. In [4], it is shown that k-bricks are the only k-DP-critical multigraphs with this property.

Theorem 1.15 fails for multigraphs, as the following example demonstrates. Fix an integer $k \in \mathbb{N}$ divisible by 3 and let G be the multigraph with vertex set $[3]$ such that $|E_G(1,2)| = k/3$ and $|E_G(1,3)| = |E_G(2,3)| = 2k/3$, so we have $2|E(G)| - k|V(G)| = k/3$. Let H be the graph with vertex set $[3] \times [3] \times [k/3]$ in which two distinct vertices (i_1, j_1, a_1) and (i_2, j_2, a_2) are adjacent if and only if one of the following three (mutually exclusive) situations occurs:

1. $\{i_1, i_2\} = [2]$ and $j_1 = j_2$;
2. $\{i_1, i_2\} \neq [2]$ and $j_1 \neq j_2$; or
3. $(i_1, j_1) = (i_2, j_2)$.

For each $i \in [3]$, let $L(i) \coloneqq \{i\} \times [3] \times [k/3]$. Then (L, H) is a cover of G and $|L(i)| = k$ for all $i \in [3]$. We claim that G is not (L, H)-colorable. Indeed, suppose that $I \in \mathcal{I}(H)$ is an (L, H)-coloring of G and for each $i \in [3]$, let $I \cap L(i) =: \{(i, j_i, a_i)\}$. By the definition of H, we have $j_1 \neq j_2$, while also $j_1 = j_3 = j_2$, which is a contradiction. It is also easy to check that G is (L, H)-critical and that it does not contain any k-brick as a subgraph.

In light of the above example, we propose the following problem:

Problem 4.2. *Let $k \geq 3$. Let G be a multigraph and let (L, H) be a cover of G such that G is (L, H)-critical and $|L(u)| = k$ for all $u \in V(G)$. Suppose that G does not contain any k-brick as a subgraph. What is the minimum possible value of the difference $2|E(G)| - k|V(G)|$, as a function of k?*

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