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Many cliques in H-free subgraphs of random graphs

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For two fixed graphs T and H let $ex(G(n,p),T,H)$ be the random variable counting the maximum number of copies of T in an H-free subgraph of the random graph $G(n,p)$. We show that for the case $T = K_m$ and $\chi(H) > m$ the behavior of $ex(G(n,p),K_m,H)$ depends strongly on the relation between p and $m_2(H) = \max_{H' \subseteq H, |V(H')|' \geq 3} \left\{ \frac{e(H')-1}{v(H')-2} \right\}$.

When $m_2(H) > m_2(K_m)$ we prove that with high probability, depending on the value of p, either one can maintain almost all copies of K_m, or it is asymptotically best to take a $\chi(H)-1$ partite subgraph of $G(n,p)$. The transition between these two behaviors occurs at $p = n^{-1/m_2(H)}$. When $m_2(H) < m_2(K_m)$ we show that the above cases still exist, however for $\delta > 0$ small at $p = n^{-1/m_2(H)+\delta}$ one can typically still keep most of the copies of K_m in an H-free subgraph of $G(n,p)$. Thus, the transition between the two behaviors in this case occurs at some p significantly bigger than $n^{-1/m_2(H)}$.

To show that the second case is not redundant we present a construction which may be of independent interest. For each $k \geq 4$ we construct a family of k-chromatic graphs $G(k,\epsilon_i)$ where $m_2(G(k,\epsilon_i))$ tends to $\frac{(k+1)(k-2)}{2(k-1)} < m_2(K_{k-1})$ as i tends to infinity. This is tight for all values of k as for any k-chromatic graph G, $m_2(G) > \frac{(k+1)(k-2)}{2(k-1)}$.

1. Introduction

The well known Turán function, denoted $ex(n,H)$, counts the maximum number of edges in an H-free subgraph of the complete graph on n vertices (see for example [22] for a survey). A natural generalization of this question is to change the base graph and instead of taking a subgraph of the complete graph consider a subgraph of a random graph. More precisely let $G(n,p)$ be the random graph on n vertices where each edge is chosen randomly and independently with probability p. Let $ex(G(n,p),H)$ denote the random variable counting the maximum number of edges in an H-free subgraph of $G(n,p)$.

The behavior of $ex(G(n,p),H)$ is studied in [8], and additional results appear in [18], [13], [11], [12] and more. Taking an extremal graph G which is H-free on n vertices with $ex(n,H)$ edges and then keeping each edge of G randomly and independently with probability p shows that w.h.p., that is, with probability tending to 1 as n tends to infinity,

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$$ex(G(n,p),H) \geq (1+o(1))ex(n,H)p.$$

In [13] Kohayakawa, Łuczak and Rödl and in [11] Haxell, Kohayakawa and Łuczak conjectured that the opposite inequality is asymptotically valid for values of p for which each edge in $G(n,p)$ takes part in a copy of H.

This conjecture was proved by Conlon and Gowers in [6], for the balanced case, and by Schacht in [20] for general graphs (see also [5] and [19]). Motivated by the condition that each edge is in a copy of H, define the *2-density* of a graph H, denoted by $m_2(H)$, to be

$$m_2(H) = \max_{H' \subseteq H, v(H') \geq 3} \left\{ \frac{e(H')-1}{v(H')-2} \right\}.$$

The Erdős-Simonovits-Stone theorem states that $ex(n,H) = \binom{n}{2}\left(1 - \frac{1}{\chi(H)-1} + o(1)\right)$, and so the theorem proved in the papers above, restated in simpler terms is the following

Theorem 1.1 ([6],[20]). *For any fixed graph H the following holds w.h.p.*

$$ex(G(n,p),H) = \begin{cases} \left(1 - \frac{1}{\chi(H)-1} + o(1)\right)\binom{n}{2}p & \text{for } p \gg n^{-1/m_2(H)} \\ (1+o(1))\binom{n}{2}p & \text{for } p \ll n^{-1/m_2(H)} \end{cases}$$

where here and in what follows we write $f(n) \gg g(n)$ when $\lim_{n\to\infty} \frac{f(n)}{g(n)} = \infty$.

Another generalization of the classical Turán question is to ask for the maximum number of copies of a graph T in an H-free subgraph of the complete graph on n vertices. This function, denoted $ex(n,T,H)$, is studied in [3] and in some special cases in the references therein. Combining both generalizations we define the following. For two graphs T and H, let $ex(G(n,p),T,H)$ be the random variable whose value is the maximum number of copies of T in an H-free subgraph of $G(n,p)$. Note that as before the expected value of $ex(G(n,p),T,H)$ is at least $ex(n,T,H)p^{e(T)}$ for any T and H.

In [3] it is shown that for any H with $\chi(H) = k > m$, $ex(n,K_m,H) = (1+o(1))\binom{k-1}{m}(\frac{n}{k-1})^m$. This motivates the following question analogous to the one answered in Theorem 1.1: *For which values of p is it true that $ex(G(n,p),K_m,H) = (1+o(1))\binom{k-1}{m}(\frac{n}{k-1})^m p^{\binom{m}{2}}$ w.h.p.?*

We show that the behavior of $ex(G(n,p),K_m,H)$ depends strongly on the relation between $m_2(K_m)$ and $m_2(H)$. When $m_2(H) > m_2(K_m)$ there are two regions in which the random variable behaves differently. If p is much smaller than $n^{-1/m_2(H)}$ then the H-free subgraph of $G \sim G(n,p)$ with the maximum number of copies of K_m has w.h.p. most of the copies of K_m in G as only a negligible number of edges take part in a copy of H. When p is much bigger than $n^{-1/m_2(H)}$ we can no longer keep most of the copies of K_m in an H-free subgraph and it is asymptotically best to take a $(k-1)$-partite subgraph of $G(n,p)$. The last part also holds when $m_2(H) = m_2(K_m)$. Our first theorem is the following:

Theorem 1.2. *Let H be a fixed graph with $\chi(H) = k > m$. If p is such that $\binom{n}{m}p^{\binom{m}{2}}$ tends to infinity as n tends to infinity then w.h.p.*

$$ex(G(n,p),K_m,H) = \begin{cases} (1+o(1))\binom{k-1}{m}(\frac{n}{k-1})^m p^{\binom{m}{2}} & \text{for } p \gg n^{-1/m_2(H)} \text{ provided } m_2(H) \geq m_2(K_m) \\ (1+o(1))\binom{n}{m}p^{\binom{m}{2}} & \text{for } p \ll n^{-1/m_2(H)} \text{ provided } m_2(H) > m_2(K_m) \end{cases}$$

Theorem 1.2 is valid when $m_2(H) > m_2(K_m)$. What about graphs H with $\chi(H) = k > m$ as before but $m_2(H) < m_2(K_m)$? Do such graphs H exist at all?

A graph H is k-critical if $\chi(H) = k$ and for any subgraph $H' \subset H$, $\chi(H') < k$. In [15] Kostochka and Yancey show that if $k \geq 4$ and H is k-critical, then

$$e(H) \geq \left\lceil \frac{(k+1)(k-2)v(H) - k(k-3)}{2(k-1)} \right\rceil.$$

This implies that for every k-critical n-vertex graph H,

$$\frac{e(H)-1}{v(H)-2} \geq \frac{(k+1)(k-2)n - k(k-3) - 2(k-1)}{2(k-1)(n-2)} > \frac{(k+1)(k-2)}{2(k-1)}. \tag{1}$$

Therefore for any H with $\chi(H) = k$ one has

$$m_2(H) > \frac{(k+1)(k-2)}{2(k-1)}.$$

This implies that Theorem 1.2 covers any graph H for which $\chi(H) \geq m+2$, since $m_2(K_m) = \frac{m+1}{2}$.

When $\chi(H) = m+1$ the situation is more complicated. Before investigating the function $ex(G(n,p), K_m, H)$ for these graphs we show that the case $m_2(H) < m_2(K_m)$ and $\chi(H) = m+1$ is not redundant. To do so we prove the following theorem, which may be of independent interest. The theorem strengthens the result in [1] for $m = 3$, expands it to any m, and by [15] it is tight.

Theorem 1.3. *For every fixed $k \geq 4$ and $\epsilon > 0$ there exist infinitely many k-chromatic graphs $G(k, \epsilon)$ with*

$$m_2(G(k,\epsilon)) \leq (1+\epsilon)\frac{(k+1)(k-2)}{2(k-1)}.$$

This theorem shows that there are infinitely many $m+1$ chromatic graphs H with $m_2(H) < m_2(K_m)$. For these graphs there are three regions of interest for the value of p: p much bigger than $n^{-1/m_2(K_m)}$, p much smaller than $n^{-1/m_2(H)}$, and p in the middle range.

One might suspect that as before the function $ex(G(n,p), K_m, H)$ will change its behavior at $p = n^{-1/m_2(H)}$ but this is no longer the case. We prove that for some graphs H when p is slightly bigger than $n^{-1/m_2(H)}$ we can still take w.h.p. an H-free subgraph of $G(n,p)$ that contains most of the copies of K_m:

Theorem 1.4. *Let H be a graph such that $\chi(H) = m+1 \geq 4$, $m_2(H) < c$ for some $c < m_2(K_m)$ and there exists $H_0 \subseteq H$ for which $\frac{e(H_0)-1}{v(H_0)-2} = m_2(H)$ and $v(H_0) > M(m,c)$ where $M(m,c)$ is large enough. If $p \leq n^{-\frac{1}{m_2(H)}+\delta}$ for $\delta := \delta(m,c) > 0$ small enough and $\binom{n}{m}p^{\binom{m}{2}}$ tends to infinity as n tends to infinity, then w.h.p.*

$$ex(G(n,p), K_m, H) = (1+o(1))\binom{n}{m}p^{\binom{m}{2}}.$$

On the other hand, we prove that for big enough values of p one cannot find an H-free subgraph of $G(n,p)$ with $(1+o(1))\binom{n}{m}p^{\binom{m}{2}}$ copies of K_m and it is asymptotically best to take a $k-1$-partite subgraph of $G(n,p)$.

As an example we show that the theorem above can be applied to the graphs constructed in Theorem 1.3.

Lemma 1.5. *For every two integers k and N there is $\epsilon > 0$ small enough such that $v(G_0(k,\epsilon)) > N$, where $G_0(k,\epsilon)$ is a subgraph of $G(k,\epsilon)$ for which $\frac{e(G_0(k,\epsilon))-1}{v(G_0(k,\epsilon))-2} = m_2(G(k,\epsilon))$.*

The rest of the paper is organized as follows. In Section 2 we establish some general results for $G(n,p)$. In Section 3 we prove Theorem 1.2. In Section 4 we describe the construction of sparse graphs with a given chromatic number and prove Theorem 1.3. In Section 5 we prove Theorem 1.4 and Lemma 1.5. We finish with some concluding remarks and open problems in Section 6.

2. Auxiliary Results

We need the following well known Chernoff bounds on the upper and lower tails of the binomial distribution (see e.g. [4], [17])

Lemma 2.1. *Let $X \sim Bin(n,p)$ then*

1. $\mathbb{P}(X < (1-a)\mathbb{E}X) < e^{\frac{-a^2\mathbb{E}X}{2}}$ *for* $0 < a < 1$
2. $\mathbb{P}(X > (1+a)\mathbb{E}X) < e^{\frac{-a^2\mathbb{E}X}{3}}$ *for* $0 < a < 1$
3. $\mathbb{P}(X > (1+a)\mathbb{E}X) < e^{\frac{-a\mathbb{E}X}{3}}$ *for* $a > 1$

The following known result is used a few times

Theorem 2.2 (see, e.g., Theorem 4.4.5 in [4])**.** *Let H be a fixed graph. For every subgraph H' of H (including H itself) let $X_{H'}$ denote the number of copies of H' in $G(n,p)$. Assume p is such that $\mathbb{E}[X_{H'}] \to \infty$ for every $H' \subseteq H$. Then w.h.p.*

$$X_H = (1+o(1))\mathbb{E}[X_H].$$

In addition we prove technical lemmas to be used in Sections 3 and 5. From here on for two graphs G and H we denote by $\mathcal{N}(G,H)$ the number of copies of H in G.

Lemma 2.3. *Let $G \sim G(n,p)$ with $p \gg n^{-1/m_2(K_m)}$ then w.h.p.*

1. *Every set of $o(pn^2)$ edges takes part in $o\big(\mathcal{N}(G,K_m)\big)$ copies of K_m,*
2. *For every $\epsilon > 0$ small enough every set of $n^{-\epsilon}pn^2$ edges takes part in at most $n^{-\epsilon/3}\mathcal{N}(G,K_m)$ copies of K_m.*

Proof. Let $G \sim G(n,p)$ and let X be the random variable counting the number of copies of K_m on a randomly chosen edge of $G(n,p)$. First we show that $\mathbb{E}[X^2] \le O(\mathbb{E}^2[X])$. Given an edge let $\{A_1,...A_l\}$ be all the possible copies of K_m using this edge in K_n and let $|A_i \cap A_j|$ be the number of vertices the copies share. Let X_{A_i} be the indicator of the event $A_i \subset G$. Then $X = \sum X_{A_i}$ and we get that

$$\mathbb{E}^2[X] = (\sum \mathbb{E}[X_{A_i}])^2 = \Theta([n^{m-2}p^{\binom{m}{2}-1}]^2)$$
$$\mathbb{E}[X^2] = \mathbb{E}[\sum_{k=2}^{m}\sum_{|A_i\cap A_j|=k} X_{A_i}X_{A_j}]$$
$$\le \sum_{k=2}^{m} n^{2m-k-2}p^{\binom{m}{2}+\binom{m-k}{2}+(m-k)k-1}$$

Put $S_k = n^{2m-k-2}p^{\binom{m}{2}+\binom{m-k}{2}+(m-k)k-1}$ and note that $S_2 = \Theta(\mathbb{E}^2[X])$. Furthermore, for any $2 < k \le m$ the following holds $S_2/S_k = n^{k-2}p^{\binom{k}{2}-1} \xrightarrow{n\to\infty} \infty$ as $p \gg n^{-1/m_2(K_m)} \ge n^{-1/m_2(K_k)}$ and from this

$$\mathbb{E}[X^2] \le O(\mathbb{E}^2[X]). \tag{2}$$

(Note that in fact $\mathbb{E}[X^2] = (1+o(1))\mathbb{E}^2[X]$ but the above estimate suffices for our purpose here)

Let $M = \mathcal{N}(G, K_m)$. To prove the first part assume towards a contradiction that there is a set of edges, $E_0 \subseteq E(G)$, which is of size $o(n^2 p)$ and that there exists $c > 0$ such that there are cM copies of K_m containing at least one edge from it.

On one hand, $\mathbb{E}^2[X] = [M\binom{m}{2}\frac{1}{e(G)}]^2$. On the other hand by Jensen's inequality

$$\mathbb{E}[X^2] \ge \mathbb{E}[X^2 \mid e \in E_0]\mathbb{P}[e \in E_0] \ge \left(\frac{cM}{|E_0|}\right)^2 \cdot \frac{|E_0|}{e(G)} = \left(\frac{M\binom{m}{2}}{e(G)}\right)^2 \frac{c^2}{\binom{m}{2}^2}\frac{e(G)}{|E_0|} = \omega(\mathbb{E}^2[X])$$

where the last equality holds as $|E_0| = o(e(G))$. This is a contradiction to (2) and so the first part of the Lemma holds.

For the second part assume there is a set E_0 such that $|E_0| = n^{-\epsilon}pn^2$ and the set of copies of K_m using edges of E_0 is of size at least $n^{-\epsilon/3}M$. Note that w.h.p. $e(G) \ge \frac{1}{4}n^2 p$. Repeating the calculation above we get that

$$\mathbb{E}[X^2] \ge \mathbb{E}[X^2 \mid e \in E_0]\mathbb{P}[e \in E_0] = (\frac{n^{-\epsilon/3}M}{n^{-\epsilon}e(G)})^2 \cdot \frac{n^{-\epsilon}}{4} = \frac{M^2}{e(G)^2}\frac{n^{\epsilon/3}}{4} = \omega(\mathbb{E}^2[X])$$

which is again a contradiction, and thus the second part of the lemma holds. □

Lemma 2.4. *Let $G \sim G(n,p)$ for $p = n^{-a}$ with $-a < -1/m_2(K_m)$. Then w.h.p. the number of copies of K_m sharing an edge with other copies of K_m is $o(n^m p^{\binom{m}{2}})$.*

Proof. First note that $n^{m-2}p^{\binom{m}{2}-1} = (np^{(m+1)/2})^{m-2} = n^{-\alpha(m-2)}$ for some $\alpha > 0$. The expected number of pairs of copies of K_m sharing a vertices, where $m-1 \ge a \ge 2$ is at most

$$\begin{aligned} n^{2m-a}p^{\binom{m}{2}+\binom{m-a}{2}+(m-a)a} &= n^m p^{\binom{m}{2}} \cdot (np^{\frac{m+a-1}{2}})^{(m-a)} \\ &< n^m p^{\binom{m}{2}} np^{\frac{m+1}{2}} \\ &= n^m p^{\binom{m}{2}} n^{-\alpha}. \end{aligned}$$

Here we used the fact that $np^{\frac{m+1}{2}} < 1$ and $p < 1$.

Using Markov's inequality we get that the probability that G has more than $2n^m p^{\binom{m}{2}} n^{-\alpha/2}$ copies of K_m sharing an edge is no more than $n^{-\alpha/2}$. □

3. Proof of Theorem 1.2

To prove Theorem 1.2, we prove three lemmas for three ranges of values of p using different approaches. Lemmas 3.1 and 3.2 are stated in a more general form as they are also used in Section 5. An explanation on how the lemmas prove Theorem 1.2 follows after the statements.

Lemma 3.1. *Let H be a fixed graph with $\chi(H) = k > m$ and let $p \gg \max\{n^{-\frac{1}{m_2(H)}}, n^{-\frac{1}{m_2(K_m)}}\}$. Then*

$$ex(G(n,p), K_m, H) = (1+o(1))\binom{k-1}{m}\left(\frac{n}{k-1}\right)^m p^{\binom{m}{2}}.$$

Lemma 3.2. *Let H be a fixed graph with $\chi(H) = k > m$, let $p < \min\{n^{-\frac{1}{m_2(H)}-\delta}, n^{-\frac{1}{m_2(K_m)}-\delta}\}$ for some fixed $\delta > 0$ and assume $n^m p^{\binom{m}{2}}$ tends to infinity as n tends to infinity. Then*

$$ex(G(n,p), K_m, H) = (1 + o(1))\binom{n}{m}p^{\binom{m}{2}}.$$

Lemma 3.3. *Let H be a fixed graph with $\chi(H) = k > m$ and let $n^{-1/m_2(K_m)-\epsilon} < p \ll n^{-1/m_2(H)}$ where $\epsilon > 0$ is sufficiently small. Then*

$$ex(G(n,p), K_m, H) = (1 + o(1))\binom{n}{m}p^{\binom{m}{2}}.$$

Lemma 3.1 takes care of the first part of Theorem 1.2. If $m_2(H) \geq m_2(K_m)$ then $n^{-1/m_2(H)} \geq n^{-1/m_2(K_m)}$ and this lemma covers values of p for which $p \gg n^{-1/m_2(H)}$.

For the second part of Theorem 1.2 we have Lemmas 3.2 and 3.3. If $m_2(H) > m_2(K_m)$ Lemma 3.2 covers values of p for which $p < n^{-1/m_2(K_m)-\delta}$ and Lemma 3.3 covers the range $n^{-1/m_2(K_m)-\epsilon} < p \ll n^{-1/m_2(H)}$. Choosing $\epsilon > \delta$ makes sure we do not miss values of p.

We mostly focus on the proof of Lemma 3.1, as the other two are simpler. Lemmas 3.1 and 3.2 are also relevant for the case $m_2(H) < m_2(K_m)$, and are used again in Section 5. For the proof of Lemma 3.1 we need several tools.

Lemma 3.4. *Let G be a k-partite complete graph with each side of size n, let $p \in [0, 1]$ and let G' be a random subgraph of G where each edge is chosen randomly and independently with probability p. If $n^m p^{\binom{m}{2}}$ goes to infinity together with n then the number of copies of K_m for $m < k$ with each vertex in a different V_i is w.h.p.*

$$(1 + o(1))\binom{k}{m}n^m p^{\binom{m}{2}}.$$

To prove the lemma, we use the following concentration result:

Lemma 3.5 (see, e.g., Corollary 4.3.5 in [4])**.** *Let $X_1, X_2, ..., X_r$ be indicator random variables for events A_i, and let $X = \sum_{i=1}^{r} X_i$. Furthermore assume $X_1, ..., X_r$ are symmetric (i.e. for every $i \neq j$ there is a measure preserving mapping of the probability space that sends event A_i to A_j). Write $i \sim j$ for $i \neq j$ if the events A_i and A_j are not independent. Set $\Delta^* = \sum_{i\sim j} \mathbb{P}(A_j|A_i)$ for some fixed i. If $\mathbb{E}[X] \to \infty$ and $\Delta^* = o(\mathbb{E}[X])$ then $X = (1 + o(1))\mathbb{E}(X)$.*

Proof of lemma 3.4. The expected number of copies of K_m in G' is $(1 + o(1))\binom{k}{m}n^m p^{\binom{m}{2}}$. So we only need to show that it is indeed concentrated around its expectation. To do so we use Lemma 3.5.

Let A_i be the event that a specific copy of K_m appears in G', and X_i be its indicator function. Clearly the number of copies of K_m in G' is $X = \sum X_i$. In this case $i \sim j$ if the corresponding copies of K_m share edges. We write $i \cap j = a$ if the two copies share exactly a vertices. It is clear that the variables X_i are symmetric. By the definition in the lemma,

$$\begin{aligned}
\Delta^* &= \sum_{i \sim j} \mathbb{P}(A_j | A_i) \\
&= \sum_{2 \le a \le m-1} \sum_{i \cap j = a} \mathbb{P}(A_j | A_i) \\
&\le \sum_{2 \le a \le m-1} \binom{m}{a} \binom{k-a}{m-a} n^{m-a} p^{\binom{m-a}{2} + (m-a)a} \\
&= o(\binom{k}{m} n^m p^{\binom{m}{2}}).
\end{aligned}$$

The last inequality holds as $n^m p^{\binom{m}{2}} = n^{m-a} p^{\binom{m-a}{2}+(m-a)a} \cdot n^a p^{\binom{a}{2}}$ and $n^a p^{\binom{a}{2}} = (np^{\frac{a-1}{2}})^a$ tends to infinity as n tends to infinity for $a < m$. □

To prove the upper bound in Lemma 3.1 we use a standard technique for estimating the number of copies of a certain graph inside another. This is done by applying Szemeredi's regularity lemma and then a relevant counting lemma. The regularity lemma allows us to find an equipartition of any graph into a constant number of sets $\{V_i\}$, such that most of the pairs of sets $\{V_i, V_j\}$ are regular (i.e. the densities between large subsets of sets V_i and V_j do not deviate by more than ϵ from the density between V_i and V_j).

In a sparse graph (such as a dense subgraph of a sparse random graph) we need a stronger definition of regularity than the one used in dense graphs. Let U and V be two disjoint subsets of $V(G)$. We say that they form an (ϵ, p)-regular pair if for any $U' \subseteq U, V' \subseteq V$ such that $|U'| \ge \epsilon |U|$ and $|V'| \ge \epsilon |V|$:

$$|d(U', V') - d(U, V)| \le \epsilon p,$$

where $d(X, Y) = \frac{|E(X,Y)|}{|X||Y|}$ is the edge density between two disjoint sets $X, Y \subseteq V(G)$.

Furthermore, an (ϵ, p)-partition of the vertex set of a graph G is an equipartition of $V(G)$ into t pairwise disjoint sets $V(G) = V_1 \cup ... \cup V_t$ in which all but at most ϵt^2 pairs of sets are (ϵ, p)-regular. For a dense graph, Szemerédi's regularity lemma assures us that we can always find a regular partition of the graph into at most $t(\epsilon)$ parts, but this is not enough for sparse graphs. For the case of subgraphs of random graphs, one can use a variation by Kohayakawa and Rödl [14] (see also [21], [2] and [16] for some related results).

In this regularity lemma we add an extra condition. We say that a graph G on n vertices is (η, p, D)-*upper-uniform* if for all disjoint sets $U_1, U_2 \subset V(G)$ such that $|U_i| > \eta n$ one has $d(U_1, U_2) \le Dp$. Given this definition we can now state the needed lemma:

Theorem 3.6 ([14])**.** *For every $\epsilon > 0$, $t_0 > 0$ and $D > 0$, there are η, T and N_0 such that for any $p \in [0, 1]$, each (η, p, D)-upper-uniform graph on $n > N_0$ vertices has an (ϵ, p)-regular partition into $t \in [t_0, T]$ parts.*

In order to estimate the number of copies of a certain graph after finding a regular partition one needs counting lemmas. We use a proposition from [7] to show that a certain cluster graph is H-free, and to give a direct estimate on the number of copies of K_m. To state the proposition we need to introduce some notation. For a graph H on k vertices, $\{1, ..., k\}$, and for a sequence of integers $\mathbf{m} = (m_{ij})_{ij \in E(H)}$, we denote by $\mathcal{G}(H, n', \mathbf{m}, \epsilon, p)$ the following family of graphs. The vertex set of each graph in the family is a disjoint union of sets $V_1, ..., V_k$ such that $|V_i| = n'$ for all i. As for the edges, for each $ij \in E(H)$ there is an (ϵ, p)-regular bipartite graph with m_{ij} edges between the sets V_i and V_j, and these are all the edges in the graph. For any $G \in \mathcal{G}(H, n', \mathbf{m}, \epsilon, p)$ denote by $G(H)$ the number of copies of H in G in which every vertex i is in the set V_i.

Proposition 3.7 ([7]). *For every graph H and every $\delta, d > 0$, there exists $\xi > 0$ with the following property. For every $\eta > 0$, there is a $C > 0$ such that if $p \geq Cn^{-1/m_2(H)}$ then w.h.p. the following holds in $G(n,p)$.*

1. *For every $n' \geq \eta n$, $\boldsymbol{m}$ with $m_{ij} \geq dp(n')^2$ for all $ij \in H$ and every subgraph G of $G(n,p)$ in $\mathcal{G}(H, n', \boldsymbol{m}, \epsilon, p)$,*

$$G(H) \geq \xi \left(\prod_{ij \in E(H)} \frac{m_{ij}}{(n')^2} \right) (n')^{v(H)}. \tag{3}$$

2. *Moreover, if H is strictly balanced, i.e. for every proper subgraph H' of H one has $m_2(H) > m_2(H')$, then*

$$G(H) = (1 \pm \delta) \left(\prod_{ij \in E(H)} \frac{m_{ij}}{(n')^2} \right) (n')^{v(H)}. \tag{4}$$

Note that the first part tells us that if G is a subgraph of $G(n,p)$ in $\mathcal{G}(H, n', \mathbf{m}, \epsilon, p)$, then it contains at least one copy of H with vertex i in V_i.

We can now proceed to the proof of Lemma 3.1, starting with a sketch of the argument. Note that the same steps can be applied to determine $ex(G(n,p), T, H)$ for graphs T and H for which $ex(n, T, H) = \Theta(n^{v(T)})$ and $p \gg \max\{n^{-1/m_2(H)}, n^{-1/m_2(T)}\}$.

Let G be an H-free subgraph of $G(n,p)$ maximizing the number of copies of K_m. First apply the sparse regularity lemma (Theorem 3.6) to G and observe using Chernoff and properties of the regular partition that there are only a few edges inside clusters and between sparse or irregular pairs. By lemma 2.3 these edges do not contribute significantly to the count of K_m. We can thus consider only graphs G which do not have such edges.

By Proposition 3.7 the cluster graph must be H-free and taking G to be maximal we can assume all pairs in the cluster graph have the maximal possible density. Applying Proposition 3.7 again to count the number of copies of K_m reduces the problem to the dense case solved in [3].

We continue with the full details of the proof.

Proof of Lemma 3.1. A $(k-1)$-partite graph with sides of size $\frac{n}{k-1}$ each is an n-vertex H-free graph containing $(1+o(1))\binom{k-1}{m}(\frac{n}{k-1})^m$ copies of K_m. We can get a random subgraph of it by keeping each edge with probability p, independently of the other edges. Then by Lemma 3.4 the number of copies of K_m in it is $(1+o(1))\binom{k-1}{m}(\frac{n}{k-1})^m p^{\binom{m}{2}}$ w.h.p., proving the required lower bound on $ex(G(n,p), K_m, H)$.

For the upper bound we need to show that no H-free subgraph of $G(n,p)$ has more than $(1+o(1))\binom{k-1}{m}(\frac{n}{k-1})^m p^{\binom{m}{2}}$ copies of K_m. Let G be an H-free subgraph of $G(n,p)$ with the maximum number copies of K_m. To use Theorem 3.6, we need to show that G is (η, p, D)-upper-uniform for some constant D, say $D = 2$, and $\eta > 0$. Indeed, taking any two disjoint subsets V_1, V_2 of size $\geq \eta n$, we get that the number of edges between them is bounded by the number of edges between them in $G(n,p)$, which is distributed like $Bin(|V_1| \cdot |V_2|, p)$. Applying Part 3 of Lemma 2.1 and the union bound gives us that w.h.p. the number of edges between any two such sets is $\leq 2|V_1| \cdot |V_2| p$ and so $d(V_1, V_2) < 2p$ as needed. Thus by Theorem 3.6, G admits an (ϵ, p)-regular partition into t parts $V(G) = V_1 \cup \cdots \cup V_t$.

Define the *cluster graph of* G to be the graph whose vertices are the sets V_i of the partition and there is an edge between two sets if the density of the bipartite graph induced by them is at least δp for some fixed small $\delta > 0$, and they form an (ϵ, p)-regular pair.

First we show that w.h.p. the cluster graph is H-free. Assume that there is a copy of H in the cluster graph, induced by the sets $V_1, \ldots, V_{v(H)}$. Consider these sets in the original graph G. To apply Part 1 of Proposition 3.7 first note that indeed $p \geq Cn^{-1/m_2(H)}$. Furthermore if $ij \in E(H)$ then by the definition of the cluster graph V_i and V_j form an (ϵ, p)-regular pair and there are at least $\delta p(\frac{n}{t})^2$ edges between them. Thus the graph spanned by the edges between $V_1, \ldots, V_{v(H)}$ in G is in $\mathcal{G}(H, \frac{n}{t}, \mathbf{m}, \epsilon, p)$ where $m_{ij} \geq \delta p(\frac{n}{t})^2$, and so w.h.p. it contains a copy of H with vertex i in the set V_i. This contradicts the fact that G was H-free to start with.

If the cluster graph is indeed H-free, as proven in [3], Proposition 2.2, since $\chi(H) > m$ then $ex(t, K_m, H) = (1 + o(1))\binom{k-1}{m}(\frac{t}{k-1})^m$. This gives a bound on the number of copies of K_m in the cluster graph. For sets $V_1, ..., V_m$ that span a copy of K_m in the cluster graph we would like to bound the number of copies of K_m with a vertex in each set in the original graph G.

To do this, we use Part 2 of Proposition 3.7. Note that we cannot use Lemma 3.4 as we need it for every subgraph of $G(n, p)$ and not only for a specific one. Part 2 can be applied only to balanced graphs, and indeed any subgraph of K_m is $K_{m'}$ for some $m' < m$ and $m_2(K_{m'}) = \frac{m'+1}{2} < \frac{m+1}{2} = m_2(K_m)$. As we would like to have a upper bound on the number of copies of K_m with a vertex in each set, we can assume that the bipartite graph between V_i and V_j has all of the edges from $G(n, p)$.

By Parts 1 and 2 of Lemma 2.1, w.h.p. for any V_i and V_j of size $\frac{n}{t}$, $|E(V_i, V_j)| = (1+o(1))p(\frac{n}{t})^2$. Thus the graph induced by the sets $V_1, ..., V_m$ in $G(n, p)$ is in $\mathcal{G}(K_m, \frac{n}{t}, \mathbf{m}, \epsilon, p)$ where $m_{ij} = (1 + o(1))p(\frac{n}{t})^2$ for any pair ij. From this the number of copies of K_m in G with a vertex in every V_i is at most $(1 + o(1))p^{\binom{m}{2}}(\frac{n}{t})^m$. Plugging this into the bound on the number of copies of K_m in the cluster graph implies that the number of copies of K_m coming from copies of K_m in the cluster is w.h.p. at most

$$(1 + o(1))\binom{k-1}{m}(\frac{t}{k-1})^m \cdot p^{\binom{m}{2}}(\frac{n}{t})^m = (1 + o(1))\binom{k-1}{m}(\frac{n}{k-1})^m \cdot p^{\binom{m}{2}}.$$

It is left to show that the number of copies of K_m coming from other parts of the graph is negligible.

To do this we show that the number of edges inside clusters and between non-dense or irregular pairs is negligible. By Chernoff (Part 3 of Lemma 2.1) the number of edges inside a cluster is at most $2p\binom{n/t}{2}t \leq 2p\frac{n^2}{t}$. The number of irregular pairs is at most ϵt^2, and again by Chernoff there are no more than $2p(\frac{n}{t})^2 \cdot \epsilon t^2 = 2\epsilon pn^2$ edges between these pairs. Finally, the number of edges between non-dense pairs is at most $\delta p(\frac{n}{t})^2 t^2 = \delta pn^2$.

As ϵ, δ and $\frac{1}{t}$ can be chosen as small as needed we get that the number of such edges is $o(n^2p)$ Thus we may apply Lemma 2.3 and conclude that the number of copies of K_m containing at least one of these edges is $o(n^m p^{\binom{m}{2}})$.

Therefore, for any H-free $G \subset G(n, p)$ the number of copies of K_m in G is at most $(1 + o(1))\binom{k-1}{m}(\frac{n}{k-1})^m p^{\binom{m}{2}}$ as needed. □

The proofs of the other two lemmas are a bit simpler.

Proof of Lemma 3.2. As $p < n^{-1/m_2(K_m)-\delta}$ we can first delete all copies of K_m sharing an edge with other copies and by Lemma 2.4 we deleted w.h.p. only $o(n^m p^{\binom{m}{2}})$ copies of K_m. Let H' be a subgraph of H for which $\frac{e(H')-1}{v(H')-1} = m_2(H)$. Let e be an edge of H' and define $\{H_i\}$ to be the family of all graphs obtained by gluing a copy of K_m to the edge e in H' and allowing any further intersection. Note that the number of graphs in $\{H_i\}$ depends only on H' and m. One can make G into an H-free graph by deleting the edge e from every copy of a graph from $\{H_i\}$ and every edge that does not take part in a copy of K_m. As we may assume every edge takes part in at most one copy of K_m it is enough to show that the number of copies of graphs from $\{H_i\}$ is $o(n^m p^{\binom{m}{2}})$.

For a fixed graph J, let X_J be the random variable counting the number of copies of J in $G \sim G(n,p)$. With this notation,

$$\mathbb{E}(X_{K_m}) = \Theta(n^m p^{\binom{m}{2}}) = \Theta(n^2 p (np^{m_2(K_m)})^{m-2}),$$
$$\mathbb{E}(X_{H_i}) = \Theta(n^2 p (np^{m_2(H_i)})^{v(H_i)-2}).$$

As $m_2(H_i) \geq m_2(K_m)$ and $p < n^{-1/m_2(K_m)}$, we get $np^{m_2(H_i)} \leq np^{m_2(K_m)} \ll 1$. Furthermore as $v(H_i) > m$ (otherwise H' would be a subgraph of K_m) we get that $(np^{m_2(H_i)})^{v(H_i)-2} = o((np^{m_2(K_m)})^{m-2})$ and thus $\mathbb{E}(X_{H_i}) = o(\mathbb{E}(X_{K_m})$.

If p is such that the expected number of copies of K_m, the graphs $\{H_i\}$ and any of their subgraphs goes to infinity as n goes to infinity we can apply Theorem 2.2 and get that $X_{K_m} = (1+o(1))\binom{n}{m}p^{\binom{m}{2}}$ and the number of copies of H_i is w.h.p. $(1+o(1))\mathbb{E}(X_{H_i}) = o(\mathbb{E}(X_{K_m}))$. Thus if we remove all edges playing the part of e in any H_i the number of copies of K_m will still be $(1+o(1))\binom{n}{m}p^{\binom{m}{2}}$.

Finally, if the number of copies of some subgraph of H_i does not tend to infinity as n tends to infinity we can remove all of the edges taking part in it, and the number of edges removed is $o(\binom{n}{m}p^{\binom{m}{2}})$. As each edge takes part in a single copy of K_m, we still get that the number of copies of K_m in this graph is $(1+o(1))\binom{n}{m}p^{\binom{m}{2}}$, as needed. □

Proof of Lemma 3.3. Let $n^{-1/m_2(K_m)-\epsilon} < p \ll n^{-1/m_2(H)}$ and $G \sim G(n,p)$. Let H' be a subgraph of H for which $\frac{e(H')-1}{v(H')-2} = m_2(H)$. We show that if G is made H-free by removing a single edge from every copy of H' then the number of copies of K_m deleted is $o(\binom{n}{m}p^{\binom{m}{2}})$. Theorem 2.2 assures us that the number of copies of K_m in G is $(1+o(1))\binom{n}{m}p^{\binom{m}{2}}$ and so it stays essentially the same after removing all copies of H'.

The expected number of copies of H' in G is

$$\mathbb{E}[\mathcal{N}(G,H')] = \Theta(n^2 p (np^{m_2(H')})^{v(H')-2}) = o(n^2 p).$$

Thus by Markov's inequality w.h.p. $\mathcal{N}(G,H') = o(n^2p)$. If $p \gg n^{-1/m_2(K_m)}$ then by Lemma 2.3 deleting all these edges removes only $o(n^m p^{\binom{m}{2}})$ copies of K_m.

As for smaller values of p, namely $p \leq O(n^{-1/m_2(K_m)})$, it follows that

$$\mathbb{E}[\mathcal{N}(G,H')] = \Theta(n^2 p (np^{m_2(H')})^{v(H')-2}) \leq n^{-\beta} n^2 p$$

for some $\beta > 0$. By Markov's inequality w.h.p. the number of edges taking part in a copy of H' in G is at most $n^{-\alpha} n^2 p$ for, say, $\alpha = \frac{\beta}{2}$.

Since $p \leq O(n^{-1/m_2(K_m)})$ Lemma 2.3 cannot be applied directly. To take care of this, define $q = n^{2\epsilon} p \gg n^{-1/m_2(K_m)}$. Lemma 2.3 applied to $G(n,q)$ implies that a set of at most $n^{-\alpha} q n^2$ edges takes part in no more than $n^{-\delta} n^m q^{\binom{m}{2}}$ copies of K_m , where $\delta = \delta(\alpha) > 0$.

The number of copies of K_m containing a member of a set of edges in $G(n,p)$ is monotone in p and in the size of the set. Thus when deleting a single edge from each copy of H' in $G(n,p)$ the number of copies of K_m removed is w.h.p. at most $n^{-\delta} n^m q^{\binom{m}{2}} = n^{-\delta + 2\binom{m}{2}\epsilon} n^m p^{\binom{m}{2}}$. Choosing ϵ small enough implies that the number of copies of K_m removed is $o(n^m p^{\binom{m}{2}})$ as needed. □

4. Construction of graphs with small 2-density

In the proof of Theorem 1.3 we construct a family of graphs $\{G(k,\epsilon)\}$ that are k-critical and $m_2(G(k,\epsilon)) = (1+\epsilon)M_k$ where M_k is the smallest possible value of m_2 for a k-chromatic graph. The following notation will be useful. For a graph G and $A \subseteq V(G)$ such that $|A| \geq 3$, let $d_G^{(2)}(A) = \frac{e(G[A])-1}{|A|-2}$. By definition, $m_2(G) = \min_{A \subseteq V(G)\,:\,|A|\geq 3} d_G^{(2)}(A)$.

Proof of Theorem 1.3. We construct the graphs $G(k,\epsilon)$ in three steps. In Step 1 we construct so called (k,t)-*towers* and derive some useful properties of them. In Step 2 we make from (k,t)-*towers* more complicated (k,t)-*complexes* and *supercomplexes*, and in Step 3 we replace each edge in a copy of K_k with a supercomplex and prove the needed.

Step 1: Towers Let $t = t(\epsilon) = \lceil k^3/\epsilon \rceil$. The (k,t)-*tower with base* $\{v_{0,0}v_{0,1}\}$ is the graph $T_{k,t}$ defined as follows. The vertex set of $T_{k,t}$ is $V_0 \cup V_1 \cup \ldots \cup V_t$, where $V_0 = \{v_{0,0}, v_{0,1}\}$ and for $1 \leq i \leq t$, $V_i = \{v_{i,0}, v_{i,1}, \ldots, v_{i,k-2}\}$. For $i = 1,\ldots,t$, $T_{k,t}[V_i]$ induces $K_{k-1} - e$ with the missing edge $v_{i,0}v_{i,1}$. Also for $i = 1,\ldots,t$, vertex $v_{i-1,0}$ is adjacent to $v_{i,j}$ for all $0 \leq j \leq (k-2)/2$ and vertex $v_{i-1,1}$ is adjacent to $v_{i,j}$ for all $(k-1)/2 \leq j \leq k-2$. There are no other edges.

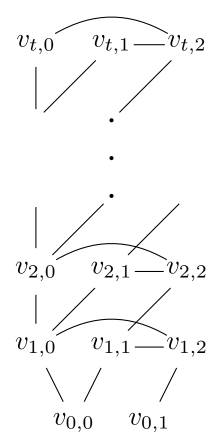

Figure 1: $T_{4,t}$

By construction, $|E(T_{k,t})| = t\left(\binom{k-1}{2} - 1 + (k-1)\right) = t\frac{(k+1)(k-2)}{2} = (|V(T_{k,t})|-2)\frac{(k+1)(k-2)}{2(k-1)}$, that is,

$$d^{(2)}_{T_{k,t}}(V(T_{k,t})) = \frac{(k+1)(k-2)}{2(k-1)} - \frac{1}{|V(T_{k,t})|-2}. \tag{5}$$

Also, since for each $i = 1,\ldots,t$, $|N(v_{i-1,1}) \cap V_i| \leq (k-1)/2$ and among the $\lceil (k-1)/2 \rceil$ neighbors of $v_{i-1,0}$ in V_i, $v_{i,0}$ and $v_{i,1}$ are not adjacent to each other,

$$\omega(T_{k,t}) = k-2. \tag{6}$$

Our first goal is to show that $T_{k,t}$ has no dense subgraphs. We will use the language of potentials to prove this. For a graph H and $A \subseteq V(H)$, let

$\rho_{k,H}(A) = (k+1)(k-2)|A| - 2(k-1)|E(H[A])|$ be the *potential of A in H*.

A convenient property of potentials is that if $|A| \geq 3$, then

$$\rho_{k,H}(A) \geq 2(k+1)(k-2) - 2(k-1) \text{ if and only if } d_H^{(2)}(A) \leq \tfrac{(k+1)(k-2)}{2(k-1)}, \tag{7}$$

but potentials are also well defined for sets with cardinality two or less.

Lemma 4.1. *Let $T = T_{k,t}$. For every $A \subseteq V(T)$,*

$$\text{if } |A| \geq 2, \text{ then } \rho_{k,T}(A) \geq 2(k+1)(k-2) - 2(k-1). \tag{8}$$

Moreover,

$$\text{if } V_0 \subseteq A, \text{ then } \rho_{k,T}(A) \geq 2(k+1)(k-2). \tag{9}$$

Proof. Suppose the lemma is not true. Among $A \subseteq V(T)$ with $|A| \geq 2$ for which (8) or (9) does not hold, choose A_0 with the smallest size. Let $a = |A_0|$.

If $a = 2$, then $\rho_{k,T}(A_0) = 2(k+1)(k-2) - 2(k-1)|E(T[A_0])| \geq 2(k+1)(k-2) - 2(k-1)$. Moreover, if $a = 2$ and $V_0 \subseteq A$, then $V_0 = A_0$ and so $E(T[A_0]) = \emptyset$. This contradicts the choice of A_0. So

$$a \geq 3. \tag{10}$$

Let i_0 be the maximum i such that $A_0 \cap V_i \neq \emptyset$. By (10), $i_0 \geq 1$. Let $A' = A_0 \cap V_{i_0}$ and $a' = |A'|$.

Case 1: $a' \leq k-2$ and $a - a' \geq 2$. Since $|(A_0 - A') \cap V_0| = |A_0 \cap V_0|$, by the minimality of a, (8) and (9) hold for $A_0 - A'$. Thus,

$$\rho_{k,T}(A_0) \geq \rho_{k,T}(A_0 - A') + a'(k+1)(k-2) - 2(k-1)\left(a' + \binom{a'}{2}\right)$$

$$= \rho_{k,T}(A_0 - A') + a'\left[(k^2 - k - 2) - 2k + 2 - (k-1)(a'-1)\right].$$

Since $k \geq 4$ and $a' \leq k-2$, the expression in the brackets is at least $k^2 - 3k - (k-1)(k-3) = k - 3 > 0$, contradicting the choice of A_0.

Case 2: $A' = V_{i_0}$ and $a - a' \geq 2$. Then $a' = k-1$. As in Case 1, (8) and (9) hold for $A_0 - A'$. Thus,

$$\rho_{k,T}(A_0) \geq \rho_{k,T}(A_0 - A') + a'(k+1)(k-2) - 2(k-1)\left(a' + \binom{a'}{2} - 1\right)$$

$$= \rho_{k,T}(A_0 - A') + (k-1)\left[(k^2 - k - 2) - 2(k-1) - (k-1)((k-1)-1) + 2\right]$$

$$\geq \rho_{k,T}(A_0 - A') + (k-1)^2\left[(k-2) - (k-2)\right] = \rho_{k,T}(A_0 - A'),$$

contradicting the minimality of A_0.

Case 3: $a = a'$, i.e., $A_0 = A'$. Then $V_0 \not\subseteq A_0$ and $a' \geq 3$. If $a \leq k-2$, then

$$\rho_{k,T}(A_0) \geq a(k+1)(k-2) - 2(k-1)\binom{a}{2} = a[(k+1)(k-2) - (k-1)(a-1)]. \tag{11}$$

Since the RHS of (11) is quadratic in a with the negative leading coefficient, it is enough to evaluate the RHS of (11) for $a = 2$ and $a = k-2$. For $a = 2$, it is $2(k+1)(k-2) - 2(k-1)$, exactly as in (8). For $a = k-2$, it is

$$(k-2)[(k+1)(k-2) - (k-1)(k-2-1)] = (3k-5)(k-2),$$

and $(3k-5)(k-2) \geq 2(k+1)(k-2) - 2(k-1)$ for $k \geq 4$. If $a = k-1$, then $A_0 = V_i$ and

$$\rho_{k,T}(A_0) = a(k+1)(k-2) - 2(k-1)\left(\binom{a}{2} - 1\right) = (k-1)((k+1)(k-2) - (k-1)(k-2) + 2)$$

$$= 2(k-1)^2 > 2(k+1)(k-2) - 2(k-1).$$

Case 4: $a - a' = 1$. As in Case 3, $V_0 \not\subseteq A_0$ and $a' \geq 2$. Let $\{z\} = A_0 - A'$. Repeating the argument of Case 3, we obtain that $\rho_{k,T}(A') \geq 2(k+1)(k-2) - 2(k-1)$. So, if $d_{T[A_0]}(z) \leq \frac{k-1}{2}$, then

$$\rho_{k,T}(A_0) \geq \rho_{k,T}(A') + (k+1)(k-2) - 2(k-1)\frac{k-1}{2} = \rho_{k,T}(A') + k - 3 > \rho_{k,T}(A'),$$

a contradiction to the choice of A_0. And the only way that $d_{T[A_0]}(z) > \frac{k-1}{2}$, is that $z = v_{i-1,0}$, k is even, and $A' \supseteq \{v_{i,0}, \ldots, v_{i,(k-2)/2}\}$. Then edge $v_{i,0}v_{i,1}$ is missing in $T[A']$, and hence

$$\rho_{k,T}(A_0) = (a'+1)(k+1)(k-2) - 2(k-1)\left(\binom{a'}{2} - 1 + k/2\right). \tag{12}$$

Since the RHS of (12) is quadratic in a' with the negative leading coefficient and $a' \geq k/2$, it is enough to evaluate the RHS of (12) for $a' = k/2$ and $a' = k-1$. For $a' = k/2$, it is

$$\frac{k+2}{2}(k+1)(k-2) - (k-1)\left(\frac{k(k-2)}{4} - 2 + k\right) = \frac{k-2}{4}(k^2 + 3k + 8).$$

Since $\frac{k^2+3k+8}{4} > 2k$ for $k \geq 4$ and $2k(k-2) > 2(k+1)(k-2) - 2(k-1)$, we satisfy (8). If $a' = k-1$, then the RHS of (12) is

$$k(k+1)(k-2) - (k-1)[(k-1)(k-2) - 2 + k] = (k-2)[k(k+1) - (k-1)^2 - k + 1]$$

$$= 2k(k-2) > 2(k+1)(k-2) - 2(k-1). \quad \square$$

Graph $T_{k,t}$ also has good coloring properties.

Lemma 4.2. *Suppose $T_{k,t}$ has a $(k-1)$-coloring f such that*

$$f(v_{0,1}) = f(v_{0,0}). \tag{13}$$

Then for every $1 \leq i \leq t$,

$$f(v_{i,1}) = f(v_{i,0}). \tag{14}$$

Proof. We prove (14) by induction on i. For $i = 0$, this is (13). Suppose (14) holds for $i = j < t$. Since $V_{j+1} \subseteq N(v_{i,0}) \cup N(v_{i,1})$, the color $f(v_{j,1}) = f(v_{j,2})$ is not used on V_{j+1} and thus $f(v_{j+1,1}) = f(v_{j+1,0})$, as claimed. $\square$

Step 2: Tower complexes A *tower complex* $C_{k,t}$ is the union of k copies $T^1_{k,t},\dots,T^k_{k,t}$ of the tower $T_{k,t}$ such that every two of them have the common base $V^1_0=\dots=V^k$, are vertex-disjoint apart from that, and have no edges between $T^i_{k,t}-V^i_0$ and $T^j_{k,t}-V^j_0$ for $j\neq i$. This common base $V^0=\{v_{0,0},v_{0,1}\}$ will be called *the base of* $C_{k,t}$.

Lemma 4.1 naturally extends to complexes as follows.

Lemma 4.3. *Let $C=C_{k,t}$. For every $A\subseteq V(C)$,*

$$\text{(15)}\qquad \text{if } |A|\geq 2, \text{ then } \rho_{k,C}(A)\geq 2(k+1)(k-2)-2(k-1).$$

Moreover,

$$\text{(16)}\qquad \text{if } A\supseteq V_0, \text{ then } \rho_{k,C}(A)\geq 2(k+1)(k-2).$$

Proof. Let $A\subseteq V(C)$ with $|A|\geq 2$, and $A_0=A\cap V_0$. Let $A_i=A\cap V(T^i_{k,t})$ if $A\cap V(T^i_{k,t})-V_0\neq\emptyset$, and $A_i=\emptyset$ otherwise. Let $I=\{i\in[t]\,:\,A_i\neq\emptyset\}$. If $|I|\leq 1$, then A is a subset of one of the towers, and we are done by Lemma 4.1. So let $|I|\geq 2$.

Case 1: $V_0\subseteq A$. Then for each nonempty A_i, $|A_i|\geq 3$ and by Lemma 4.1, $\rho_{k,C}(A_i)\geq 2(k+1)(k-2)$. So, by the definition of the potential,

$$\rho_{k,C}(A)=\sum_{i\in I}\rho_{k,C}(A_i)-(|I|-1)2(k+1)(k-2)\geq |I|2(k+1)(k-2)-(|I|-1)2(k+1)(k-2)=2(k+1)(k-2).$$

Case 2: $V_0\cap A=\{v_{0,j}\}$, where $j\in\{1,2\}$. Then for each nonempty A_i, $|A_i|\geq 2$ and by Lemma 4.1, $\rho_{k,C}(A_i)\geq 2(k+1)(k-2)-2(k-1)$. So, by the definition of the potential and the fact that $|I|\geq 2$,

$$\rho_{k,C}(A)=\sum_{i\in I}\rho_{k,C}(A_i)-(k+1)(k-2)(|I|-1)$$

$$\geq |I|(2(k+1)(k-2)-2(k-1))-(k+1)(k-2)(|I|-1)=|I|((k+1)(k-2)-2(k-1))+(k+1)(k-2)$$

$$\geq 2((k+1)(k-2)-2(k-1))+(k+1)(k-2)>2(k+1)(k-2)-2(k-1),$$

when $k\geq 4$.

Case 3: $V_0\cap A=\emptyset$. Then $\rho_{k,C}(A)=\sum_{i\in I}\rho_{k,C}(A_i)$. Since $\rho_{k,C}(A_i)\geq (k+1)(k-2)$ for every $i\in I$ and $|I|\geq 2$, $\rho_{k,C}(A)\geq 2(k+1)(k-2)$, as claimed. □

Given a tower complex $C_{k,t}$, let $W_0=\{v^1_{t,0},\dots,v^k_{t,0}\}$ and $W_1=\{v^1_{t,1},\dots,v^k_{t,1}\}$. Then the auxiliary *bridge graph* $B_{k,t}$ is the bipartite graph with parts W_0 and W_1 whose edges are defined as follows. For each pair (i,j) with $1\leq i<j\leq k$, if $j-i\leq k/2$, then $B_{k,t}$ contains edge $v^i_{t,0}v^j_{t,1}$, otherwise it contains edge $v^i_{t,1}v^j_{t,0}$. There are no other edges.

By construction, $B_{k,t}$ has exactly $\binom{k}{2}$ edges, and the maximum degree of $B_{k,t}$ is $\lfloor k/2\rfloor$. It is important that

$$\text{(17)}\qquad \text{for each } 1\leq i<j\leq k, \text{ an edge in } B_{k,t} \text{ connects } \{v^i_{t,0},v^i_{t,1}\} \text{ with } \{v^j_{t,0},v^j_{t,1}\}.$$

The *supercomplex* $S_{k,t}$ is obtained from a tower complex $C_{k,t}$ by adding to it all edges of $B_{k,t}$. The main properties of $S_{k,t}$ are stated in the next three lemmas.

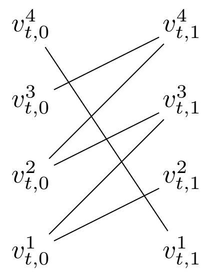

Figure 2: $B_{4,t}$

Lemma 4.4. *For each $(k-1)$-coloring f of $S_{k,t}$,*

$$f(v_{0,1}) \neq f(v_{0,0}). \tag{18}$$

Proof. Suppose $S_{k,t}$ has a $(k-1)$-coloring f with $f(v_{0,1}) = f(v_{0,0})$. Then by Lemma 4.2, $f(v^i_{t,1}) = f(v^i_{t,0})$ for every $1 \leq i \leq k$. Thus by (17), the k colors $f(v^1_{t,0}), f(v^2_{t,0}), \dots, f(v^k_{t,0})$ are all distinct, a contradiction. □

Lemma 4.5. *Let $S = S_{k,t}$ with base V_0. For every $A \subseteq V(S) - V_0$,*

$$\text{if } |A| \geq 2, \text{ then } \rho_{k,S}(A) \geq 2(k+1)(k-2) - 2(k-1). \tag{19}$$

Proof. Suppose the lemma is not true. Let C be the copy of $C_{k,t}$ from which we obtained S by adding the edges of $B = B_{k,t}$. Among $A \subseteq V(S) - V_0$ with $|A| \geq 2$ and $\rho_{k,S}(A) < 2(k+1)(k-2) - 2(k-1)$, choose A_0 with the smallest size. Let $a = |A_0|$. Let $I = \{i \in [t] : A_0 \cap V(T^i_{k,t}) \neq \emptyset\}$. If $|I| \leq 1$, then A is a subset of one of the towers, and we are done by Lemma 4.1. So let $|I| \geq 2$.

If $a = 2$, then

$$\rho_{k,S}(A_0) = a(k+1)(k-2) - 2(k-1)|E(S[A_0])| \geq 2(k+1)(k-2) - 2(k-1),$$

contradicting the choice of A_0. So $a \geq 3$. Furthermore, if $a = 3$, then since $|I| \geq 2$, $B_{k,t}$ is bipartite, and $v^i_{t,0}v^i_{t,1} \notin E(S)$ for any i, the graph $S[A_0]$ has at most two edges and so $\rho_{k,S}(A_0) \geq 3(k+1)(k-2) - 2(2(k-1)) > 2(k+1)(k-2) - 2(k-1)$. Thus

$$a \geq 4. \tag{20}$$

If $d_{S[A_0]}(w) \leq \frac{k-1}{2}$ for some $w \in A_0$, then

$$\rho_{k,S}(A_0 - w) \leq \rho_{k,S}(A_0) - (k+1)(k-2) + \frac{k-1}{2}2(k-1) = \rho_{k,S}(A_0) + 3 - k < \rho_{k,S}(A_0).$$

By (20), this contradicts the minimality of a. So,

$$\delta(S[A_0]) \geq \tfrac{k}{2}. \text{ In particular, } a \geq 1 + \tfrac{k}{2}. \tag{21}$$

Let $E(A_0, B)$ denote the set of edges of B both ends of which are in A_0. Then since $A_0 \cap V_0 = \emptyset$,

$$\rho_{k,S}(A_0) = \rho_{k,C}(A_0) - 2(k-1)|E(A_0, B)| = \sum_{i \in I} \rho_{k,C}(A_i) - 2(k-1)|E(A_0, B)|. \tag{22}$$

Let $I_1 = \{i \in I : |A_0 \cap V(T^i_{k,t})| = 1\}$ and $I_2 = I - I_1$. By Lemma 4.1, for each $i \in I_2$, $\rho_{k,S}(A_i) \geq 2(k+1)(k-2)-2(k-1)$. Thus if $I_1 = \emptyset$, then by (22) and the fact that $|E(A_0, B)| \leq \binom{|I|}{2}$, we have

$$\rho_{k,S}(A_0) \geq |I|(2(k+1)(k-2) - 2(k-1)) - \binom{|I|}{2} 2(k-1) = |I|(2k^2 - 3k - 3 - |I|(k-1)).$$

The minimum of the last expression is achieved either for $|I| = 2$ or for $|I| = k$. If $|I| = 2$, this is $2(2k^2 - 5k - 1) > 2(k+1)(k-2) - 2(k-1)$. If $|I| = k$, this is $k(k^2 - 2k - 3)$, which is again greater than $2(k+1)(k-2) - 2(k-1)$. Thus $|I_1| \neq \emptyset$.

Suppose $i, i' \in I_1$, $w \in A_i$, $w' \in A_{i'}$ and $ww' \in E(S)$. Let $A' = A_0 - w - w'$. By the definition of I_1, all edges of $S[A_0]$ incident with w or w' are in $E(B)$. Since $\Delta(B) \leq \frac{k}{2}$, $|E(S[A_0])| - |E(S[A'])| \leq k-1$. Thus

$$\rho_{k,S}(A') \leq \rho_{k,S}(A_0) - 2(k+1)(k-2) + (k-1)2(k-1) = \rho_{k,S}(A_0) - 2k + 6.$$

But by (20), $|A'| \geq 2$, a contradiction to the minimality of a. It follows that for every $i \in I_1$, each neighbor in A_0 of the vertex $w \in A_i$ is in some A_j for $j \in I_2$. This implies $|E(A_0, B)| \leq \binom{|I|}{2} - \binom{|I_1|}{2}$. Together with (21) and $\Delta(B) = \lfloor k/2 \rfloor$, this yields that for each $i \in I_1$, the vertex $w \in A_i$ has exactly $k/2$ neighbors in B, and all these neighbors are in A. In particular, $|I_2| \geq \frac{k}{2}$ and k is even. Moreover, if $i, i' \in I_1$, $w \in A_i$ and $w' \in A_{i'}$, then their neighborhoods in B are distinct, and thus in this case $|I_2| > \frac{k}{2}$. Since k is even, this implies

$$|I_2| \geq \frac{k+2}{2}. \tag{23}$$

Since the potential of a single vertex is $(k+1)(k-2)$,

$$\rho_{k,S}(A_0) \geq |I|(2(k+1)(k-2)-2(k-1)) - |I_1|((k+1)(k-2)-2(k-1)) - \left(\binom{|I|}{2} - \binom{|I_1|}{2}\right) 2(k-1). \tag{24}$$

The expression $-|I_1|((k+1)(k-2) - 2(k-1) + \binom{|I_1|}{2} 2(k-1)$ in (24) decreases when $|I_1|$ grows but is at most $\frac{k-2}{2}$. Thus by (23), it is enough to let $|I_1| = |I| - \frac{k+2}{2}$ in (24). So,

$$\rho_{k,S}(A_0) \geq |I|(k+1)(k-2) + \frac{k+2}{2}((k+1)(k-2) - 2(k-1)) - (k-1)(k+2)(|I| - \frac{k+4}{4})$$

$$= -2k|I| + \frac{k+2}{2}\left[k^2 - 3k + \frac{k^2+3k-4}{2}\right] \geq -2k^2 + \frac{(k+2)(3k^2-3k-4)}{4} > 2(k+1)(k-2) - 2(k-1)$$

for $k \geq 4$. □

Lemma 4.6. *Let $S = S_{k,t}$ with base V_0. Let $A \subseteq V(S)$ and $|A| \leq t+1$.*

$$\text{If } |A| \geq 2, \text{ then } \rho_{k,S}(A) \geq 2(k+1)(k-2) - 2(k-1). \tag{25}$$

Moreover,

$$\text{if } A \supseteq V_0, \text{ then } \rho_{k,S}(A) \geq 2(k+1)(k-2). \tag{26}$$

Proof. Suppose the lemma is not true. Among $A \subseteq V(S)$ with $|A| \geq 2$ for which (25) or (26) does not hold, choose A_0 with the smallest size. Let $a = |A_0|$. By Lemma 4.3, $S[A_0]$ contains an edge ww' in B. By Lemma 4.5, A_0 contains a vertex $v \in V_0$. In particular, $a \geq 3$.

If $S[A_0]$ is disconnected, then A_0 is the disjoint union of nonempty A' and A'' such that S has no edges connecting A' with A''. Since $a \geq 3$, we may assume that $|A'| \geq 2$. By the minimality of A_0, $\rho_{k,S}(A') \geq 2(k+1)(k-2) - 2(k-1)$. Also, $\rho_{k,S}(A'') \geq (k+1)(k-2)$. Thus

$$\rho_{k,S}(A_0) = \rho_{k,S}(A') + \rho_{k,S}(A'') \geq 2(k+1)(k-2) - 2(k-1) + (k+1)(k-2) > 2(k+1)(k-2),$$

contradicting the choice of A_0. Therefore, $S[A_0]$ is connected.

Since the distance in S between $v \in V_0$ and $\{w, w'\} \subset V(B)$ is at least t, $a \geq t+2$, a contradiction. □

Step 3: Completing the construction Let $G = G(k, \epsilon)$ be obtained from a copy H of K_k by replacing every edge uv in H by a copy $S(uv)$ of $S_{k,t}$ with base $\{u, v\}$ so that all other vertices in these graphs are distinct. Suppose G has a $(k-1)$-coloring f. Since $|V(H)| = k$, for some distinct $u, v \in V(H)$, $f(u) = f(v)$. This contradicts Lemma 4.4. Thus $\chi(G) \geq k$.

Suppose there exists $A \subseteq V(G)$ with

$$|A| \geq 2 \text{ and } |E(G[A])| > 1 + (1+\epsilon)\tfrac{(k+1)(k-2)}{2(k-1)}(|A| - 2). \tag{27}$$

Choose a smallest $A_0 \subseteq V(G)$ satisfying (27) and let $a = |A_0|$. Since a 2-vertex (simple) graph has at most one edge, $a \geq 3$. We claim that

$$G[A_0] \text{ is 2-connected.} \tag{28}$$

Indeed, if not, then since $a \geq 3$, there are $x \in A_0$ and subsets A_1, A_2 of A_0 such that $A_1 \cap A_2 = \{x\}$, $A_1 \cup A_2 = A_0$, $|A_1| \geq 2$, $|A_2| \geq 2$, and there are no edges between $A_1 - x$ and $A_2 - x$ (this includes the case that $G[A_0]$ is disconnected). By the minimality of a, $|E(G[A_j])| \leq 1 + (1+\epsilon)\frac{(k+1)(k-2)}{2(k-1)}(|A_j| - 2)$ for $j = 1, 2$. So,

$$|E(G[A_0])| = |E(G[A_1])| + |E(G[A_2])| \leq 2 + (1+\epsilon)\frac{(k+1)(k-2)}{2(k-1)}(|A_1| + |A_2| - 4)$$

$$= 2 + (1+\epsilon)\frac{(k+1)(k-2)}{2(k-1)}(a-3) \leq 1 + (1+\epsilon)\frac{(k+1)(k-2)}{2(k-1)}(a-2),$$

contradicting (27). This proves (28).

Let $J = \{uv \in E(H) : A_0 \cap (V(S(uv) - u - v) \neq \emptyset\}$. For $uv \in J$, let $A_{uv} = A_0 \cap (V(S(uv))$. Since $G[A_0]$ is 2-connected, for each $uv \in J$,

$$\{u, v\} \subset A_{uv} \text{ and } G[A_{uv}] \text{ is connected. In particular, } |A_{uv}| \geq 4. \tag{29}$$

Our next claim is that for each $uv \in J$,

$$|E(G[A_{uv}])| \leq (1+\epsilon)\tfrac{(k+1)(k-2)}{2(k-1)}(|A_{uv}| - 2). \tag{30}$$

Indeed, if $|A_{uv}| \leq t+1$, this follows from Lemma 4.6. If $|A_{uv}| \geq t+2$, then by the part of Lemma 4.3 dealing with $A \supseteq V_0$,

$$|E(G[A_{uv}])| \leq |E(B_{k,t})| + \frac{(k+1)(k-2)}{2(k-1)}(|A_{uv}| - 2) = \binom{k}{2} + \frac{(k+1)(k-2)}{2(k-1)}(|A_{uv}| - 2).$$

But since $t \geq k^3/\epsilon$, $\binom{k}{2} < \epsilon t^{\frac{(k+1)(k-2)}{2(k-1)}}$. This proves (30).

By (30),

$$|E(G[A_0])| = \sum_{uv \in J} |E(G[A_{uv}])| \leq (1+\epsilon)\frac{(k+1)(k-2)}{2(k-1)} \sum_{uv \in J}(|A_{uv}| - 2) \tag{31}$$

Since each A_{uv} has at most two vertices in common with the union of all other $A_{u'v'}$, $\sum_{uv\in J}(|A_{uv}| - 2) \leq a-2$. Thus (31) contradicts the choice of A_0. It follows that no $A \subseteq V(G)$ satisfies (27), which exactly means that $m_2(G) \leq (1+\epsilon)\frac{(k+1)(k-2)}{2(k-1)}$. □

5. The case $m_2(H) < m_2(K_m)$

When $m_2(H) < m_2(K_m)$ we show that as in the previous case there are two typical behaviors of the function $ex(G(n,p), K_m, H)$. For small values of p Lemma 3.2 shows that there exists w.h.p. an H-free subgraph of $G(n,p)$ which contains all but a negligible part of the copies of K_m. For large values of p Lemma 3.1 shows that w.h.p. every H-free graph will have to contain a much smaller proportion of the copies of K_m.

However, unlike in the case $m_2(H) > m_2(K_m)$ discussed in Section 3, the change between the behaviors for $p = n^{-a}$ does not happen at $-a = -1/m_2(H)$. Theorem 1.4 shows that if $p = n^{-a}$ and $-a$ is slightly bigger than $-1/m_2(H)$ we can still take all but a negligible number of copies of K_m into an H-free subgraph. As for a conjecture about where the change happens (and if there are indeed two regions of different behavior and not more) see the discussion in the last section.

Proof of Theorem 1.4. Let $G \sim G(n,p)$ with $p = n^{-a}$ where $-a = -c + \delta$ for some small $\delta > 0$ to be chosen later. Let G' be the graph obtained from G by first removing all pairs of copies of K_m sharing an edge and then removing all edges that do not take part in a copy of K_m. As δ is small, we may assume that $-a < -1/m_2(K_m)$, apply Lemma 2.4 and deduce that w.h.p. the number of copies of K_m removed in the first step is $o(\binom{n}{m}p^{\binom{m}{2}})$. In the second step there are no copies of K_m removed, and thus w.h.p. $\mathcal{N}(G, K_m) = (1+o(1))\mathcal{N}(G', K_m)$. Furthermore, if there is a copy of H_0 in G' then each edge of it must be contained in a copy of K_m and not in two or more such copies.

Let $\mathcal{H}_m$ be the family of the following graphs. Every graph in $\mathcal{H}_m$ is an edge disjoint union of copies of K_m, it contains a copy of H_0 and removing any copy of K_m makes it H_0-free. Note that if G is $\mathcal{H}_m$-free then G' is H_0-free.

To show that G is indeed $\mathcal{H}_m$-free w.h.p. we prove that for any $H' \in \mathcal{H}_m$ the expected number of copies of it in G is $o(1)$. We will show this for $p = n^{-\frac{1}{m_2(H)}+\delta}$, and it will thus clearly hold for smaller values of p as well. For every H' the expected number of copies of it in $G(n,p)$ is $\Theta(p^{e(H')}n^{v(H')}) = \Theta(n^{-\frac{1}{m_2(H)}e(H')+v(H')}n^{\delta \cdot e(H')})$ and we want to show that it is equal $o(1)$ for any H'. For this it is enough to show that $-\frac{e(H')}{v(H')} + m_2(H) - \delta\frac{e(H')}{v(H')}m_2(H) < 0$. We first prove that

$$d(H') := \frac{e(H')}{v(H')} > m_2(H) + \delta'$$

for some $\delta' := \delta'(m,c)$ and then to finish show that $\frac{e(H')}{v(H')}m_2(H) \leq g(m)$ for some function g.

Note that every $H' \in \mathcal{H}_m$ contains a copy of H_0 and that H_0 itself does not contain a copy of K_m as $m_2(H_0) < m_2(K_m)$. The vertices of copies of K_m in H' can be either all from H_0 or use some external vertices. Let E_1 be the edges between two vertices of H_0 that are not part of the original H_0 and let $|E_1| = e_1$. Furthermore, let $V_1 \cup ... \cup V_k = V(H') \setminus V(H_0)$ be the external vertices, where each V_i creates a copy of K_m with the other vertices from H_0 and let $|V_i| = v_i$.

Each edge in H_0 must be a part of a copy of K_m. An edge in E_1 takes care of at most $\binom{m}{2}-1$ edges from H_0, and each V_i takes care of at most $\binom{m-v_i}{2}$ edges. From this we get that

$$\begin{aligned} e(H_0) \leq & \sum_{i=1}^{k} \binom{m-v_i}{2} + e_1(\binom{m}{2}-1) \\ \leq & k\binom{m-1}{2} + e_1(\binom{m}{2}-1) \\ \leq & \frac{m^2}{2}(k+e_1). \end{aligned}$$

We will take care of two cases, either $e_1 \geq \frac{e(H_0)}{m^2}$ or $k \geq \frac{e(H_0)}{m^2}$. In the first case let H_1 be the graph H_0 together with the edges in E_1. Then

$$\frac{e(H_1)}{v(H_1)} = \frac{e(H_0)+e_1}{v(H_0)} \geq (1+\frac{1}{m^2})\frac{e(H_0)}{v(H_0)}.$$

We can assume $v(H_0)$ is large enough so that $\frac{e(H_0)}{v(H_0)}/\frac{e(H_0)-1}{v(H_0)-2} \geq (1-\frac{1}{2m^2})$ and as $m_2(H_0)$ is bounded from below by a function of m, we get that for some $\delta' := \delta'(m)$ small enough we get

$$\frac{e(H_1)}{v(H_1)} \geq m_2(H_0) + \delta'.$$

Hence w.h.p. there is no copy of H_1 in G, and thus no copy of H'.

Now let us assume that $k \geq \frac{e(H_0)}{m^2}$ and let $\gamma = m_2(K_m) - m_2(H) \geq m_2(K_m) - c$. The expression $\frac{\binom{v_i}{2}+v_i(m-v_i)}{v_i}$ decreases with v_i, and as V_i creates a copy of K_m with an edge of H_0, we get that $v_i \leq m-2$ and so $\frac{\binom{v_i}{2}+v_i(m-v_i)}{v_i} \geq \frac{\binom{m}{2}-1}{m-2}$. It follows that

$$\sum_{i=0}^{k} \binom{v_i}{2} + v_i(m-v_i) \geq \sum_{i=0}^{k} v_i \frac{\binom{m}{2}-1}{m-2} = \sum_{i=0}^{k} v_i(m_2(H_0)+\gamma). \tag{32}$$

Every set of vertices V_i uses at least one edge in H_0 for a copy of K_m, and as there are no two copies of K_m sharing an edge, it follows that:

$$v(H') = v(H_0) + \sum_{i=0}^{k} v_i \leq e(H_0) + (m-1)e(H_0) = m \cdot e(H_o).$$

Combining this with the assumption on k we conclude

$$\sum_{i=0}^{k} v_i \geq k \geq \frac{e(H_0)}{m^2} \geq \frac{v(H')}{m^3}. \tag{33}$$

Finally a direct calculation yields

$$e(H_0) + e_1 > e(H_0) - 1 = \frac{e(H_0)-1}{v(H_0)-2}(v(H_0)-2) = m_2(H_0)(v(H_0)-2). \tag{34}$$

Applying the above inequalities we get

$$\begin{aligned}
e(H') =& e(H_0) + e_1 + \sum_{i=0}^{k} \binom{v_i}{2} + v_i(m - v_i) \\
\overset{32,34}{\geq}& m_2(H)(\sum_{i=0}^{k} v_i + v(H_0) - 2) + \sum_{i=0}^{k} v_i\gamma \\
=& m_2(H)(v(H') - 2) + \sum_{i=0}^{k} v_i\gamma \\
\overset{33}{\geq}& m_2(H)(v(H') - 2) + \frac{v(H')}{m^3}\gamma \\
\geq& v(H')(m_2(H_0) + \frac{1}{2m^3}\gamma).
\end{aligned}$$

The last inequality holds if $2m_2(H) \leq v(H')\frac{\gamma}{2m^3}$, but this is true as $v(H_0)$ is large enough. Thus, for $\delta' := \delta'(m, c)$ small enough,

$$\frac{e(H')}{v(H')} \geq m_2(H_0) + \frac{1}{2m^3}\gamma \geq m_2(H_0) + \frac{1}{2m^3}(m_2(K_m) - c) \geq m_2(H_0) + \delta'$$

and again, w.h.p. G will not have a copy of H'.

It is left to show that indeed $\frac{e(H')}{v(H')}m_2(H) \leq g(m)$. By the definition of H' we get that $\frac{e(H')}{v(H')} < \frac{e(H_0)+(m-2)e(H_0)}{v(H')} = (m-1)\frac{e(H_0)}{v(H_0)}$. As we may assume that $v(H_0)$ is large, it follows that $\frac{e(H_0)}{v(H_0)} \leq m_2(H_0)(1 + \frac{1}{m})$, and as $m_2(H) < m_2(K_m)$, we conclude that for some $g(m)$ the needed inequality holds.

□

To finish this section, we show that indeed the theorem can be applied to $G(m + 1, \epsilon)$.

Proof of Lemma 1.5. To prove this we will use the following fact. If $\frac{a}{b}$ and $\frac{p}{q}$ are rational numbers such that $0 < |\frac{a}{b} - \frac{p}{q}| \leq \frac{1}{bM}$ then $p \geq M$. Indeed, assume towards a contradiction that $q < M$, but then $|\frac{a}{b} - \frac{p}{q}| = |\frac{aq-bp}{bq}| \geq \frac{1}{bq} > \frac{1}{bM}$.

Let $G_0 := G_0(m + 1, \epsilon)$, and take $\frac{a}{b} = \frac{(m+2)(m-1)}{2m}$ and $\frac{p}{q} = \frac{e(G_0)-1}{v(G_0)-2}$. By Theorem 1.3 it follows that $|\frac{a}{b} - \frac{p}{q}| \leq \epsilon\frac{(m+2)(m-1)}{2m}$. Choosing ϵ small enough will make $v(G_0)$ as large as needed.

□

6. Concluding remarks and open problems

- It is interesting to note that there are two main behaviors of the function $ex(G(n, p), K_m, H)$ that we know of. For K_m and H with $\chi(H) = k > m$ for small p one gets that an H-free subgraph of $G \sim G(n, p)$ can contain w.h.p. most of the copies of K_m in the original G. On the other hand, when $p > \max\{n^{-1/m_2(H)}, n^{-1/m_2(K_m)}\}$ then an H-free graph with the maximal number of K_ms is essentially w.h.p. $k - 1$ partite, thus has a constant proportion less copies of K_m than G.
  If $m_2(H) > m_2(K_m)$ then Theorem 1.2 shows that the behavior changes at $p = n^{-1/m_2(H)}$, but if $m_2(H) < m_2(K_m)$ the critical value of p is bounded away from $n^{-1/m_2(H)}$ and it is not clear where exactly it is.

Looking at the graph $G \sim G(n,p)$ and taking only edges that take part in a copy of K_m yields another random graph $G|_{K_m}$. The probability of an edge to take part in $G|_{K_m}$ is $\Theta(p \cdot n^{m-2}p^{\binom{m}{2}-1})$. A natural conjecture is that if $n^{m-2}p^{\binom{m}{2}}$ is much bigger than $n^{-1/m_2(H)}$ then when maximizing the number of K_m in an H-free subgraph we cannot avoid a copy of H by deleting a negligible number of copies of K_m and when $n^{m-2}p^{\binom{m}{2}}$ is much smaller than $n^{-1/m_2(H)}$ we can keep most of the copies of K_m in an H-free subgraph of $G \sim G(n,p)$. It would be interesting to decide if this is indeed the case.

- Another possible model of a random graph, tailored specifically to ensure that each edge lies in a copy of K_m, is the following. Each m-subset of a set of n labeled vertices, randomly and independently, is taken as an m-clique with probability $p(n)$. In this model the resulting random graph G is equal to its subgraph $G|_{K_m}$ defined in the previous paragraph, and one can study the behavior of the maximum possible number of copies of K_m in an H-free subgraph of it for all admissible values of $p(n)$.
- There are other graphs T and H for which $ex(n,T,H)$ is known, and one can study the behavior of $ex(G(n,p),T,H)$ in these cases. For example in [10] and independently in [9] it is shown that $ex(n,C_5,K_3) = (n/5)^5$ when n is divisible by 5.
  Using some of the techniques in this paper we can prove that for $p \gg n^{-1/2} = n^{-1/m_2(K_3)}$, $ex(n,C_5,K_3) = (1+o(1))(np/5)^5$ w.h.p. whereas if $p \ll n^{-1/2}$ then w.h.p. $ex(n,C_5,K_3) = (\frac{1}{10}+o(1))(np)^5$. Similar results can be proved in additional cases for which $ex(n,T,H) = \Omega(n^t)$ where t is the number of vertices of T. As observed in [3], these are exactly all pairs of graphs T,H where H is not a subgraph of any blowup of T.
- When investigating $ex(G(n,p),T,H)$ here we focused on the case that T is a complete graph. It is possible that a variation of Theorem 1.2 can be proved for any T and H satisfying $m_2(T) > m_2(H)$, even without knowing the exact value of $ex(n,T,H)$.
- In the cases studied here for non-critical values of p, $ex(G(n,p),T,H)$ is always either almost all copies of T in $G(n,p)$ or $(1+o(1))ex(n,T,H)p^{e(T)}$. It would be interesting to decide if such a phenomenon holds for all T, H.
- As with the classical Turán problem, the question studied here can be investigated for a general graph T and finite or infinite families $\mathcal{H}$.

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