AN ALGORITHMIC ANSWER TO THE ORE-TYPE VERSION OF DIRAC'S QUESTION ON DISJOINT CYCLES IN MULTIGRAPHS

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Abstract. For the NP-complete problem on the existence of k disjoint cycles in an n-vertex graph G, Corrádi and Hajnal in 1963 gave sufficient conditions: For all $k \geq 1$ and $n \geq 3k$, every (simple) n-vertex graph G with minimum degree $\delta(G) \geq 2k$ contains k disjoint cycles. The same year, Dirac described the 3-connected multigraphs not containing two disjoint cycles and asked the more general question: Which $(2k-1)$-connected multigraphs do not contain k disjoint cycles? Recently, Kierstead, Kostochka and Yeager resolved this question. In this paper, we sharpen this result by presenting a description that can be checked in polynomial time of all multigraphs G with no k disjoint cycles for which the underlying simple graph $\underline{G}$ satisfies the following Ore-type condition: $d_{\underline{G}}(v)+d_{\underline{G}}(u) \geq 4k-3$ for all nonadjacent $u, v \in V(G)$.

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Dedicated to Gregory Gutin on the occasion of his 60th Birthday

1. Introduction

For a multigraph $G = (V, E)$, let $|G| = |V|$, $\|G\| = |E|$, $\delta(G)$ be the minimum degree of G, and $\alpha(G)$ be the independence number of G. For a simple graph G, let $\overline{G}$ denote the complement of G. For multigraphs G and H, let $G \cup H$ denote the multigraph with $V(G \cup H) = V(G) \cup V(H)$ and $E(G \cup H) = E(G) \cup E(H)$. For disjoint graphs G and H, let $G \vee H$ denote $G \cup H$ together with all edges from $V(G)$ to $V(H)$.

Let $K(X)$ be the complete graph with vertex set X, and $K_t(X) = K(X)$ indicate that $|X| = t$.

The problem of finding the maximum number of disjoint cycles in a graph is NP-hard, since even some partial cases of it are:

Theorem 1 ([7], p. 68). *Determining whether a $3n$-vertex graph has n disjoint triangles is an NP-complete problem.*

On the other hand, Bodlaender [1] and independently Downey and Fellows [5] showed that this problem is *fixed parameter tractable*:

Theorem 2 ([1, 5]). *For every fixed k, the question whether an n-vertex graph has k disjoint can be resolved in linear (in n) time.*

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Since the general problem is hard, it is natural to look for sufficient conditions that ensure the existence of "many" disjoint cycles in a graph. One of well-known results of this type is the following theorem of Corrádi and Hajnal [2] from 1963:

Theorem 3 ([2])**.** *Let $k \in \mathbb{Z}^+$. Every graph G with $|G| \geq 3k$ and $\delta(G) \geq 2k$ contains k disjoint cycles.*

The hypothesis $\delta(G) \geq 2k$ is best possible, as shown by the $3k$-vertex graph $H = \overline{K}_{k+1} \vee K_{2k-1}$, which has $\delta(H) = 2k - 1$ but does not contain k disjoint cycles. The proof yields a polynomial algorithm for finding k disjoint cycles in the graphs satifying the conditions of the theorem.

Theorem 3 was refined and generalized in several directions. Enomoto [6] and Wang [16] generalized the Corrádi-Hajnal Theorem in terms of the minimum Ore-degree $\sigma_2(G) := \min\{d(x) + d(y) : xy \notin E(G)\}$:

Theorem 4 ([6],[16])**.** *Let $k \in \mathbb{Z}^+$. Every graph G with (i) $|G| \geq 3k$ and*

$$\sigma_2(G) \geq 4k - 1 \tag{1.1}$$

contains k disjoint cycles.

Kierstead, Kostochka and Yeager [11] refined Theorem 3 by characterizing all simple graphs that fulfill the weaker hypothesis $\delta(G) \geq 2k-1$ and contain k disjoint cycles. This refinement depends on an extremal graph $\mathbf{Y_{k,k,k}}$ where $\mathbf{Y_{h,s,t}} = \overline{K_h} \vee (K_s \cup K_t)$ and $\mathbf{Y_{h,s,t}}(X_0, X_1, X_2) = \overline{K_h}(X_0) \vee (K_s(X_1) \cup K_t(X_2))$.

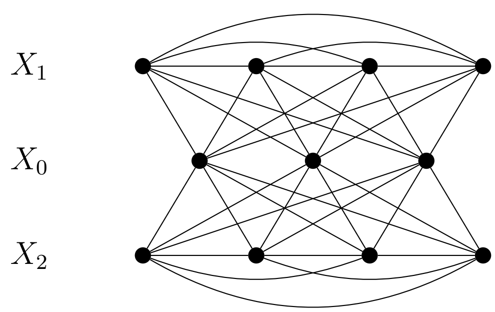


FIGURE 1.1. $\mathbf{Y_{h,t,s}}$, shown with $h = 3$ and $t = s = 4$.

Theorem 5 ([11])**.** *Let $k \geq 2$. Every simple graph G with $|G| \geq 3k$ and $\delta(G) \geq 2k - 1$ contains k disjoint cycles if and only if:*

(i) $\alpha(G) \leq |G| - 2k$;
(ii) *if k is odd and $|G| = 3k$, then $G \neq \mathbf{Y_{k,k,k}}$; and*
(iii) *if $k = 2$ then G is not a wheel.*

Theorem 4 was refined in a similar way in [11] and [10] (see Theorem 16 in the next section).

Dirac [3] described all 3-connected multigraphs that do not have two disjoint cycles and posed the following question:

Question 6 ([3]). *Which $(2k-1)$-connected multigraphs*[1] *do not have k disjoint cycles?*

Kierstead, Kostochka and Yeager [12] used Theorem 5 to answer Question 6 (see Theorem 14 in Section 2). The goal of this paper is to resolve the Ore-type version of Question 6 for multigraphs in an algorithmic way. In Theorem 17 we describe *all multigraphs G that do not have k disjoint cycles and for any two nonadjacent vertices x and y in the underlying simple graph $\underline{G}$, we have $d_{\underline{G}}(x)+d_{\underline{G}}(y) \geq 4k-3$.* Using this description we construct a polynomial time algorithm that for every multigraph satisfying the conditions of Theorem 17 either finds k disjoint cycles or shows that there are no such k cycles.

In the next section, we introduce notation and discuss existing results to be used later on. In Section 3 we state our main results, Theorem 17 and Theorem 18 . In the next four sections, we prove Theorem 17, and in the last section prove Theorem 18 .

2. Preliminaries and known results

2.1. **Notation.** For every multigraph G, let $V_1 = V_1(G)$ be the set of vertices in G incident to loops, and $V_2 = V_2(G)$ be the set of vertices in $G - V_1$ incident to strong edges. Let $F = F(G)$ be the simple graph with $V(F) = V_2$ formed by the multiple edges in $G - V_1$. We will call the edges of $F(G)$ *the strong edges of* G, and define $\alpha' = \alpha'(F)$ to be the size of a maximum matching in F. Let $\underline{G}$ denote the *underlying simple graph of* G, i.e. the simple graph on $V(G)$ such that two vertices are adjacent in G if and only if they are adjacent in $\underline{G}$. Let G^* denote the result of making all edges of G strong. For $e \notin E(G)$, let $G+e$ denote the graph with $V(G+e) = V(G)$ and $E(G+e) = E(G) \cup \{e\}$. For a path $P \in \{P_1, P_2\}$ with $P \cap G = \emptyset$, let $\mathrm{sd}(G, e, P)$ be the result of subdividing e with P.

Recall that $K_t(X) = K(X)$ denotes the complete with vertex set X where $|X| = t$. If we only want to specify one vertex v of K_t we write $K_t(v)$. Similarly, $K(Y, Z)$ is the complete Y, Z-bigraph. We also extend this notation to the case that Y is a graph. Then $K(Y, Z)$ is $K(V(Y), Z) \cup Y$.

A set $S = \{v_0, \ldots, v_s\}$ of vertices in a graph H is a *superstar with center v_0 in H* if $N_H(v_i) = \{v_0\}$ for each $1 \leq i \leq s$ and $H - S$ has a perfect matching. For a maximum matching M, set $W = W(M) = V(M)$, $V' = V'(M) = V \smallsetminus W$, and $G' = G'(M) = G[V'(M)]$. If $|F| = 2\alpha'$ then $G'(M) = G'(M')$ for all perfect matchings M and M'.

For $v \in V$, we define $s(v) = |N(v)|$ to be the *simple degree* of v, and we say that $\mathcal{S}(G) = \min\{s(v) : v \in V\}$ is the *minimum simple degree* of G. Similarly, $\mathcal{SO}(G) = \min\{s(v) + s(u) : v, u \in V, v \neq u \text{ and } uv \notin E(\underline{G})\}$. Let $c(G)$ be the maximum number of disjoint cycles contained in G.

We define $\mathcal{D}_k$ to be the family of multigraphs G with $\mathcal{S}(G) \geq 2k-1$ and $\mathcal{DO}_k$ to be the family of multigraphs G with $\mathcal{SO}(G) \geq 4k-3$. For a graph $G \in \mathcal{DO}_k$, call a vertex $v \in V(G)$ *low* if $d_G(v) \leq 2k-2$. Let $\mathcal{D}_k^0$ be the set of simple graphs in $\mathcal{D}_k$. Let $\mathcal{B}_k = \{G \in \mathcal{D}_k : c(G) < k\}$, $\mathcal{B}_k^0 = \mathcal{D}_k^0 \cap \mathcal{B}_k$, $\mathcal{B}_k^0(e)$ be the set of graphs in $\mathcal{B}_k$ whose only strong edge is e. Let $\mathcal{BO}_k = \{G \in \mathcal{DO}_k : c(G) < k\}$ and $\mathcal{BO}_k^0$ be the set of simple graphs in $\mathcal{B}O_k$.

If $G \in \mathcal{DO}_k$ is an n-vertex multigraph and $\alpha(G) \geq n-2k+2$, then for any distinct v_1, v_2 in a maximum independent set I, $s(v_1) + s(v_2) \leq (2k-2) + (2k-2) < 4k-3$. Thus $\alpha(G) \leq n-2k+1$ for every n-vertex $G \in \mathcal{DO}_k$; so we call $G \in \mathcal{DO}_k$ *extremal* if $\alpha(G) = n-2k+1$. If $G \in \mathcal{DO}_k$ is extremal, and v_1 and v_2 are distinct vertices in a

[1]Dirac used the word *graphs*, but in [3] this appears to mean *multigraphs*.

maximum independent set I, then $s(v_1)+s(v_2) \le (2k-1)+(2k-1) = 4k-2$. Since $\mathcal{S}O(G) \ge 4k-3$, this means that for some $v \in \{v_1, v_2\}$ we have $s(v) = 2k-1$ and I is exactly $V(G) - N(v)$. Thus to check whether G is extremal it is enough to check for every $v \in V(G)$ with $s(v) = 2k-1$ whether the set $V(G) - N(v)$ is independent.

A *big set* in an extremal $G \in \mathcal{DO}_k$ is an independent set of size $\alpha(G)$. If I is a big set in an extremal $G \in \mathcal{DO}_k$, then since $\mathcal{S}O(G) \ge 4k-3$, each but one vertex $v \in I$ is adjacent to each $w \in V(G)-I$, and one vertex in I may be not adjacent to one vertex in $V(G)-I$. On the other hand, if x is a common vertex of big sets I and J, then $s(x) \le |G| - |I \cup J| \le 2k-1-|J-I|$. Hence for every $y \in I - x$, $s(x)+s(y) \le 4k-2-|J-I|$, and so $|J-I| \le 1$. Furthermore, if $|J-I| = 1$ and there is $x' \in J \cap I - x$, then $s(x)+s(x') \le 2(n-\alpha(G)-1) = 4k-4$, a contradiction. Thus in this case $\alpha(G) = 2$. This yields the following.

(2.1) *Let G be extremal. If $|G| > 2k+1$ then every two big sets in G are disjoint. If $|G| = 2k+1$, sets $I, J \subset V(G)$ are big and $x \in I \cap J$, then $s(x) = 2k-2$.*

2.2. **Gallai-Edmonds Theorem.** We will use the classical Gallai-Edmonds Theorem on the structure of graphs without perfect matchings. Recall that a graph F is *odd* if $|F|$ is odd, and that $o(F)$ denotes the number of odd components of F. For a graph F and $S \subseteq V(F)$, the *deficiency* $\mathrm{def}(S)$ is $o(F-S) - |S|$. Next, $\mathrm{def}(F) := \max\{\mathrm{def}(S) : S \subseteq V(F)\}$. For each graph F, $\mathrm{def}(F) \ge 0$, since $\mathrm{def}(\emptyset) = o(F) \ge 0$.

Theorem 7 (Gallai-Edmonds)**.** *Let F be a graph and D be the set of $v \in V(F)$ such that there is a maximum matching in F not covering v. Let A be the set of the vertices in $V(F) - D$ that have neighbors in D, and let $C = V(F) - D - A$. Let $F_1, \ldots, F_k$ be the components of $F[D]$. If M is a maximum matching in F, then all of the following hold:*
a) M covers C and matches A into distinct components of $F[D]$.
b) Each F_i is factor-critical and has a near-perfect matching in M.
c) If $\emptyset \ne S \subseteq A$, then $N(S)$ intersects at least $|S|+1$ components of $F[D]$.
d) $\mathrm{def}(F) = \mathrm{def}(A) = k - |A|$.

We refer to (D, A, C) as the Gallai-Edmonds decomposition (GE-decomposition) of F.

2.3. **Results for $\mathcal{D}_k$.** Since every cycle in a simple graph has at least 3 vertices, the condition $|G| \ge 3k$ is necessary in Theorem 3. However, it is not necessary for multigraphs, since loops and multiple edges form cycles with fewer than three vertices. Theorem 3 can easily be extended to multigraphs, although the statement is no longer as simple:

Theorem 8. *For $k \in \mathbb{Z}^+$, let G be a multigraph with $\mathcal{S}(G) \ge 2k$, and set $F = F(G)$ and $\alpha' = \alpha'(F)$. Then G has no k disjoint cycles if and only if*

$$|V(G)| - |V_1(G)| - 2\alpha' < 3(k - |V_1| - \alpha'), \tag{2.2}$$

i.e., $|V(G)| + 2|V_1| + \alpha' < 3k$.

Proof. If (2.2) holds, then G does not have enough vertices to contain k disjoint cycles. If (2.2) fails, then we choose $|V_1|$ cycles of length one and α' cycles of length two from $V_1 \cup V(F)$. By Theorem 3, the remaining (simple) graph contains $k - |V_1| - \alpha'$ disjoint cycles. □

Theorem 8 yields the following.

Corollary 9. *Let G be a multigraph with $\mathcal{S}(G) \geq 2k-1$ for some integer $k \geq 2$, and set $F = F(G)$ and $\alpha' = \alpha'(F)$. Suppose G contains at least one loop. Then G has no k disjoint cycles if and only if $|V(G)| + 2|V_1| + \alpha' < 3k$.*

Since acyclic graphs are exactly forests, Theorem 5 can be restated as follows:

Theorem 10. *For $k \in \mathbb{Z}^+$, let G be a simple graph in $\mathcal{D}_k$. Then G has no k disjoint cycles if and only if one of the following holds:*
(α) $|G| \leq 3k-1$;
(β) $k = 1$ *and G is a forest with no isolated vertices;*
(γ) $k = 2$ *and G is a wheel;*
(δ) $\alpha(G) = n - 2k + 1$; *or*
(ϵ) $k > 1$ *is odd and* $G = \mathbf{Y_{k,k,k}}$.

Dirac [3] described all multigraphs in $\mathcal{D}_2$ that do not have two disjoint cycles:

Theorem 11 ([3])**.** *Let G be a 3-connected multigraph. Then G has no two disjoint cycles if and only if one of the following holds:*
(A) $\underline{G} = K_4$ and the strong edges in G form either a star (possibly empty) or a 3-cycle;
(B) $G = K_5$;
(C) $\underline{G} = K_5 - e$ and the strong edges in G are not incident to the ends of e;
(D) $\underline{G}$ is a wheel, where some spokes could be strong edges; or
(E) G is obtained from $K_{3,|G|-3}$ by adding non-loop edges between the vertices of the (first) 3-class.

Going further, Lovász [14] described *all* multigraphs with no two disjoint cycles. To state his result, let a *bud* be a vertex incident to at most one edge. Also, let $W_n = K_1 \vee C_n$ be the wheel and $\mathbf{W_n^+} = W_n \cup K(V(K_1), V(C))$ be the wheel with strong edges for spokes. Similarly, let $\mathbf{K^+_{3,n-3}} = K_3 \vee \overline{K}_{n-3}$ be the n-vertex multigraph obtained from $K_{3,n-3}$ by adding strong edges connecting all pairs of the vertices of the (first) 3-class. Then, each multigraph described by Theorem 11(A) above is contained either in $\mathbf{W_3^+}$ or in $\mathbf{K^+_{3,1}}$.

Lovász [14] observed that any connected multigraph can be transformed into a multigraph with minimum degree at least 3 or a multigraph with exactly one vertex without affecting the maximum number of disjoint cycles in it by using a sequence of operations of the following two types: (i) deleting a bud; (ii) replacing a vertex v of degree 2 that has neighbors x and y (where $v \notin \{x, y\}$ but possibly $x = y$) by a new (possibly parallel) edge connecting x and y. He also proved the following:

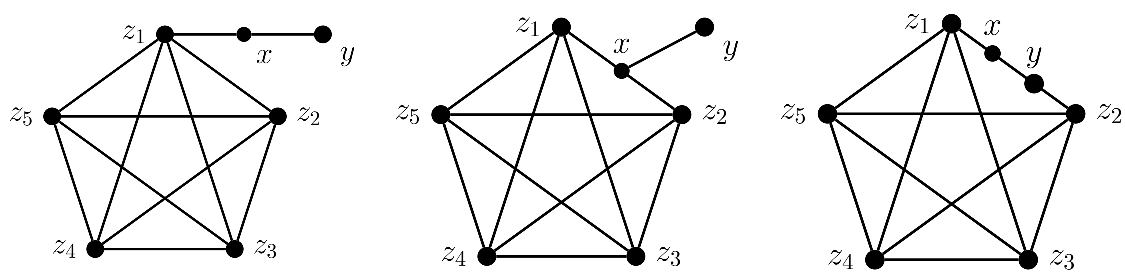
Theorem 12 ([14])**.** *Let H be a multigraph with $\delta(H) \geq 3$. Then H has no two disjoint cycles if and only if :*
- $(L1)$ $H = K_5$;
- $(L2)$ $H \subseteq \mathbf{W^+_{|G|-1}}$;
- $(L3)$ $H \subseteq \mathbf{K^+_{3,|G|-3}}$; *or*
- $(L4)$ *H is obtained from a forest T and vertex x with possibly some loops at x by adding edges linking x to T.*

Say that a multigraph G *has a 2-property* if the vertices of degree at most 2 form a clique $Q(G)$ (possibly with some multiple edges). Let $G \in \mathcal{DO}_2$ with no two disjoint cycles. Then G has a 2-property. By Lovász's observation above, G can be transformed to a multigraph

H that has exactly one vertex or is of type (L1)–(L4) by a sequence of deleting buds and/or contracting edges. Note that if a multigraph G' has 2-property, then the multigraph obtained from G' by deleting a bud or contracting an edge also has. Thus, H and all the intermediate multigraphs have 2-property. Reversing this transformation, G can be obtained from H by adding buds and subdividing edges. If H has exactly one vertex and at most one edge, then any multigraph with 2-property that can be obtained from H this way has maximum degree at most 2. Hence G is either a K_i for $i \leq 3$ or forms a strong edge. If $\delta(H) \geq 3$, then the clique $Q := Q(G)$ cannot have more than 2 vertices: by the definition of $Q(G)$, $|Q| \leq 3$, and if $|Q| = 3$ then Q induces a K_3-component of G and $\delta(G - Q) \geq 3$; thus $G - Q$ has another cycle. Let $Q' := V(G) \smallsetminus V(H)$. By above, $Q \subseteq Q'$. If $Q' \neq Q$, then Q consists of a single leaf in G with a neighbor of degree 3, so G is obtained from H by subdividing an edge and adding a leaf to the degree-2 vertex. If $Q' = Q$, then Q is a component of G, or $G = H + Q + e$ for some edge $e \in E(H, Q)$, or at least one vertex of Q subdivides an edge $e \in E(H)$. In the last case, when $|Q| = 2$, e is subdivided twice by Q.

In case (L4), because $\delta(H) \geq 3$, either T has at least two buds, each linked to x by multiple edges, or T has one bud linked to x by an edge of multiplicity at least 3. So this case cannot arise from G. Also, $\delta(H) = 3$, unless $H = K_5$, in which case $\delta(H) = 4$. So Q is not an isolated vertex, lest deleting Q leave H with $\delta(H) \geq 5 > 4$; and if Q has a vertex of degree 1 then $H = K_5$. Else all vertices of Q have degree 2, and Q consists of the subdivision vertices of one edge of H. This yields the following characterization of multigraphs in $G \in \mathcal{DO}_2$ with no two disjoint cycles.

Set $Z_t = \{z_1, \dots, z_t\}$, and define $\mathbf{S_3} = K(Z_5) \cup z_1xy$, $\mathbf{S_4} = \mathrm{sd}(K(Z_5), z_1z_2, x) \cup xy$, and $\mathbf{S_5} = \mathrm{sd}(K(Z_5), z_1z_2, xy)$ (See Figure 2.1).



(a) Graph $\mathbf{S_3}$ (b) Graph $\mathbf{S_4}$ (c) Graph $\mathbf{S_5}$

FIGURE 2.1. Graphs $\mathbf{S_3}, \mathbf{S_4}$, and $\mathbf{S_5}$

Theorem 13. *All $G \in \mathcal{BO}_2$ satisfy one of:*

(Y1) $G \subseteq \mathbf{S_3}$, the graph obtained from K_5 by attaching a new subdivided edge;

(Y2) $G \subseteq \mathbf{S_4} = \mathrm{sd}(K_5, e, x) + y + xy$;

(Y3) $G = \mathrm{sd}(K_5, e, xy)$;

(Y4) $G \subseteq H'$, where $H = \mathbf{W}^+_{|\mathbf{H}|-\mathbf{1}}$ and $H' \in \{H, \mathrm{sd}(H, e, x), \mathrm{sd}(H, e, xy)\}$;

(Y5) $G \subseteq H'$, where $H = \mathbf{K}^+_{\mathbf{3},|\mathbf{H}|-\mathbf{3}}$ and $H' \in \{H, \mathrm{sd}(H, e, x), \mathrm{sd}(H, e, xy)\}$.

By Corollary 9, in order to describe the multigraphs in $\mathcal{D}_k$ not containing k disjoint cycles, it is enough to describe such multigraphs with no loops. Recently, Kierstead, Kostochka, and Yeager [12] proved the following:

Theorem 14 ([12])**.** *Let $k \geq 2$ and $n \geq k$ be integers. Let G be an n-vertex graph in $\mathcal{D}_k$ with no loops. Set $F = F(G)$, $\alpha' = \alpha'(F)$, and $k' = k - \alpha'$. Then G does not contain k disjoint cycles if and only if one of the following holds:*

(a) $n + \alpha' < 3k$;
(b) $|F| = 2\alpha'$ (i.e., F has a perfect matching) and either
(i) k' is odd and $G - F = \mathbf{Y_{k',k',k'}}$, or
(ii) $k' = 2 < k$ and $G - F = W_5$;
(c) G is extremal and either
(i) some big set is not incident to any strong edge, or
(ii) for some two distinct big sets I_j and $I_{j'}$, all strong edges intersecting $I_j \cup I_{j'}$ have a common vertex outside of $I_j \cup I_{j'}$ and if $v \in I_j \cap I_{j'}$ (this may happen only if $k' = 2$), then v is not incident with a strong edge;
(d) $n = 2\alpha' + 3k'$, k' is odd, and F has a superstar $S = \{v_0, \ldots, v_s\}$ with center v_0 such that either
(i) $G - (F - S + v_0) = \mathbf{Y_{k'+1,k',k'}}$, or
(ii) $s = 2$, $v_1v_2 \in E(G)$, $G - F = \mathbf{Y_{k'-1,k',k'}}$ and G has no edges between $\{v_1, v_2\}$ and the set X_0 in $G - F$;
(e) $k = 2$ and $W_{n-1} \subseteq G \subseteq W^*_{n-1}$;
(f) $k' = 2$, $|F| = 2\alpha' + 1 = n - 5$, and $G - F = C_5$.

2.4. **Results for $\mathcal{DO}_k$.** Theorem 4 can be restated as follows.

Theorem 15. *For $k \in \mathbb{Z}^+$, let G be a simple graph with $\mathcal{SO}(G) \geq 4k - 1$ and $|G| \geq 3k$. Then G has k disjoint cycles.*

Theorem 12 implies a description of graphs in $\mathcal{DO}_2$ with no two disjoint cycles. To state it, we need some notation.

The next theorem summarizes the results of [11] and [10].

Theorem 16. *For $k, n \in \mathbb{Z}^+$ with $n \geq 3k$, let G be an n-vertex simple graph in $\mathcal{DO}_k$. Then G has no k disjoint cycles if and only if one of the following holds:*
(S1) $k = 1$ and G is a forest with at most one isolated vertex;
(S2) $k = 2$ and and G satisfies the conditions of Theorem 13;
(S3) $\alpha(G) = n - 2k + 1$;
(S4) $k = 3$ and $G = \mathbf{F_1}$ (see Fig. 2.2);
(S5) $k = 3$ and $G = \mathbf{F_2}$ where $\mathbf{F_2}$ is obtained from the complement $\mathbf{F_2'}$ of the graph $\mathbf{O_5}$ (see Fig. 3.1) by adding an all-adjacent vertex;
(S6) $k = 3$ and G is the graph $\mathbf{F_3}$ in Fig. 3.2;
(S7) $k \geq 3$, $n = 3k$, $\alpha(G) \leq k$, and $\chi(\overline{G}) > k$;
(S8) $k \geq 3, n = 3k$, and $G \subseteq \mathbf{Y_{k,s,2k-s}}$ for some odd $1 \leq s \leq 2k - 1$;
(S9) $k \geq 3, n = 3k$, and $G = \mathbf{Y_{k-1,1,2k}}$.

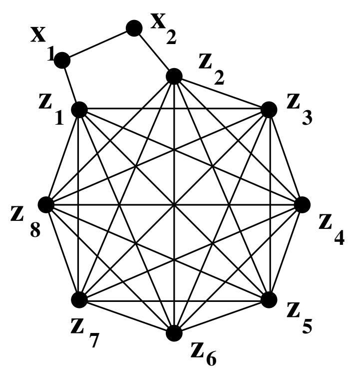


FIGURE 2.2. Graph $\mathbf{F_1}$.

Remark. The result of Rabern [15] (see also [9, 13]) implies that if (S7) holds then $k \leq 4$.

3. MAIN RESULTS

Our first main result describes the loopless multigraphs in $\mathcal{DO}_k$ with no k disjoint cycles. Our second main result uses this description to construct a polynomial-time algorithm that for every $G \in \mathcal{DO}_k$ either finds k disjoint cycles in G or proves that G has no k such cycles .

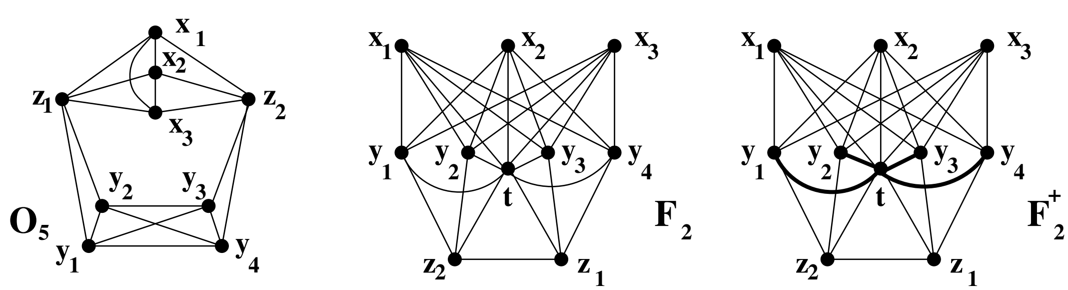


FIGURE 3.1. Graphs $\mathbf{O_5}$ and $\mathbf{F_2}$ and multigraph $\mathbf{F_2^+}$.

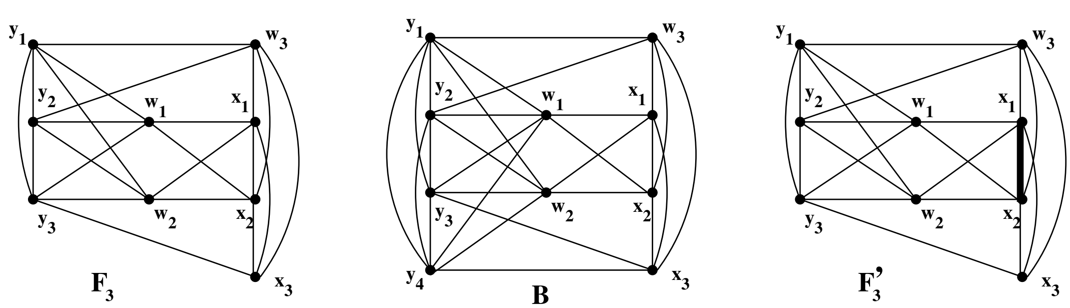


FIGURE 3.2. Graphs $\mathbf{F_3}$ and $\mathbf{B}$ and multigraph $\mathbf{F_3'}$.

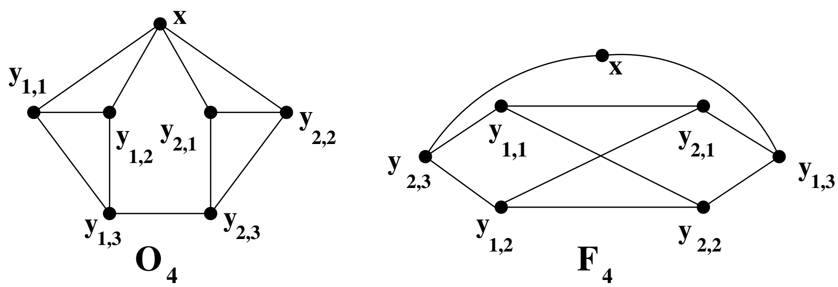


FIGURE 3.3. Graphs $\mathbf{O_4}$ and $\mathbf{F_4}$.

Theorem 17. *Let $k \geq 5$ and $n \geq k$ be integers. Let G be an n-vertex multigraph in $\mathcal{DO}_k$ with no loops. Set $F = F(G)$, $\alpha' = \alpha'(F)$, and $k' = k - \alpha'$. Let (D, A, C) be the GE-decomposition of $V(F)$ and let $D' = V(G) - V(F)$. If G does not contain k disjoint cycles then one of the following holds:*

(Q1) $n < 3k - \alpha'$;

(Q2) $3k - \alpha' \leq n \leq 3k - \alpha' + 1$, $|F| = 2\alpha'$ *(i.e., F has a perfect matching) and either*

- *(Q2a)* $G - F$ *is one of the graphs described in (S6)–(S9) of Theorem 16 with* k' *in place* k, *or*
- *(Q2b)* $2 \leq k' \leq 3$.

(Q3) $n > 2k + 1$, *G is extremal and either*

- *(Q3a) some big set is not incident to any strong edge, or*
- *(Q3b) for some two distinct big sets J and J', all strong edges intersecting $J \cup J'$ have a common vertex outside of $J \cup J'$, and any vertex $x \in J \cap J'$ (if exists) has no strong neighbors;*

(Q4) $n = 3k - \alpha' + 1$, $|D'| = 9$ *and* $|F| - 2\alpha' \in \{1, 3\}$;

(Q5) $n = 3k - \alpha'$, $k' \leq 4$ *and* $n' = 3k'$;

(Q6) $n = 3k - \alpha'$, $|D'| = 7$ *and* $|F| - 2\alpha' = 2$;

(Q7) $n = 2k + 1$ *and* $k' = 1$.

(Q8) $n > 2k + 1$, $n = 2\alpha' + 3k' = 3k - \alpha'$, *and* $\alpha' \leq 1 + (|A| + |C|)/2$.

(Q9) $n = 3k - \alpha'$, *and G has a vertex $x \in D'$ of degree $k + \alpha' - 1$ such that for each maximum matching M in F, the set $N(x) - V(M)$ is independent, and F has a maximum matching M^* such that $V(M^*) \subset N[x]$;*

(Q10) $n \geq 3k - \alpha'$, $\alpha(G) \leq n - 2k$, $k' = 2$, *and either* $n' = 6$ *or all of* $n' = 7$, $|F| = 2\alpha'$ *and* $G' = F_4$.

Theorem 18. *There is a polynomial time algorithm that for every multigraph $G \in \mathcal{DO}_k$ either finds k disjoint cycles in G or shows that G has no k disjoint cycles.*

4. PROOF OF THEOREM 17 : SIMPLER CASES

Suppose G does not have k disjoint cycles and that none of (Q1)–(Q10) holds.

Among the maximum matchings in F, choose a matching M such that

(i) $\alpha(G - W)$ minimum, where $W = V(M)$ and

(ii) modulo (i), the sum of simple degrees of the multigraph $G - W$ is maximum.

Then $|M| = \alpha'$, $G' := G - W$ is simple, and $\mathcal{SO}(G') \geq 4k - 3 - 2\alpha' = 4k' - 3$. So $G' \in \mathcal{DO}_{k'}$. Let $n' := |V(G')| = n - 2\alpha'$.

If $|G'| = 3k'$, then G' is quite dense, so sometimes it will be convenient to consider the complement of $\underline{G}$. For $v \in V(G)$, let $\overline{N}(v) = V(G) - N[v]$ and $\overline{d}(v) = |\overline{N}(v)| = n - 1 - s(v)$. When $|G'| = 3k'$, we have $n = 2k + k'$ and thus the inequality $d(v) + d(u) \geq 4k - 3$ can be written as

$$\overline{d}(v) + \overline{d}(u) \leq 2k' + 1 \quad \text{for all } vu \notin E(G). \tag{4.1}$$

Since G' has no k' disjoint cycles, either $n' < 3k'$ or one of (S1)–(S9) in Theorem 16 holds for G' with k' in place of k. If $n' < 3k'$, then (Q1) holds. So suppose $n' \geq 3k'$.

The following observation will be sometimes helpful.

Lemma 19. *If $u \in D - V(G')$, then F has a maximum matching M' and G' has a vertex w such that $M \cup M'$ has a component that is a w,u-path in F and every other component of $M \cup M'$ is a single edge. In particular, the set of vertices of G not covered by M' is $V(G') - w + u$.*

Proof. By the definition of D, F has a maximum matching M_1 not covering u. Consider $M \cup M_1$. Every component of it is a single edge or an even cycle or a path of an even length. Since u is not covered by M, it is an end of a path P in $M \cup M_1$. The other end, say w, of P must be not covered by M, i.e., $w \in V(G')$. Furthermore, the intermediate vertices in P are not in $V(G')$, since they are covered by M. Let M' be obtained from M by switching the edges along the alternating path P. Then M' satisfies the lemma. □

CASE 1: $n > 2k + 1$ and (S3) holds for G', i.e. $\alpha(G') = n' - 2k' + 1$. So G' is extremal. Let J be a big set in G'. Then $|J| = n' - 2k' + 1 = n - 2k + 1 \geq 3$. So G is extremal and J is a big set in G. If (Q3a) fails then some $w \in J$ has a strong neighbor v. Let vu be the edge in M containing v. In F, consider the maximum matching $M' = M - vu + wv$, and set $G'' = G - V(M')$. By the choice of M, G'' contains a big set J', and J' is big in G. Since $w \notin J'$ and $n - 2k + 1 \geq 3$, (2.1) implies $J' \cap J = \emptyset$ (possibly, $u \in J'$). If (Q3b) fails then there is a strong edge xy such that $x \in J \cup J'$ and $y \neq v$. Moreover, by the symmetry between J and J', we may assume $x \in J'$. Let yz be the edge in M containing y. Since M is maximum, $z \neq u$. Let $M'' = M' - yz + xy$. Again by the case, $G - V(M'')$ contains a big set J''. Similarly to above, since $w, x \notin J''$ and $n > 2k + 1$, (2.1) implies that J'' is disjoint from $J \cup J'$. So $n' \geq 3|J|$. But $n' \geq 3k'$ and thus $|J| = n' - 2k' + 1 \geq n' - 2n'/3 + 1$, a contradiction.

CASE 2: (S4) holds for G', i.e. $k' = 3$ and $G' = \mathbf{F_1}$ (see Fig. 2.2). Since for $i = 1, 2$ and $1 \leq j \leq 8$, $i \neq j$, $x_i z_j \notin E(G')$ and $d_{G'}(x_i) + d_{G'}(z_j) = 9 = 4k' - 3$,

$$\text{each vertex of } G' \text{ is adjacent in } G \text{ to each vertex in } V(M). \tag{4.2}$$

If some $vu \in M$ is such that v has a strong neighbor $z_j \in V(G') - x_1 - x_2$, then by (4.2), $ux_1, ux_2 \in E(G)$. Then the $k - k' - 1$ 2-cycles in $M - uv$ together with cycles vz_jv, ux_1x_2u and two disjoint 3-cycles in $G' - x_1 - x_2 - z_j$ form k disjoint cycles in G. Similarly, if some $vu \in M$ is such that v has a strong neighbor $x_i \in V(G')$, say vx_1 is a strong edge, then by (4.2), $ux_2, uz_2 \in E(G)$. So the $k - k' - 1$ 2-cycles in $M - uv$ together with cycles vx_1v, ux_2z_2u, $z_3z_4z_5z_3$ and $z_6z_7z_8z_6$ form k disjoint cycles in G. Thus (Q2)(b) holds.

CASE 3: (S5) holds for G', i.e. $k' = 3$ and $G' = \mathbf{F_2}$ which is obtained from the complement $\mathbf{F_2'}$ of $\mathbf{O_5}$ by adding a vertex t adjacent to all vertices in $\mathbf{F_2'}$ (see Fig. 3.1). Since each of the vertices $x_1, x_2, x_3, y_1, \ldots, y_4$ has degree 5 in $\mathbf{F_2}$ and is not adjacent to z_1 or z_2 of degree 4, and since $5 + 4 = 4k' - 3$, similarly to (4.2) we get

(4.3) each vertex of $G' - t$ is adjacent in G to each vertex in $V(M)$.

Suppose some $vu \in M$ is such that v has a strong neighbor $w \in V(G') - t$. Then we find k disjoint cycles in G as follows. Certainly, we include into the set all $k - k' - 1$ 2-cycles in $M - uv$ and the 2-cycle vwv. The remaining $k' = 3$ cycles will depend on the choice of w. By symmetry, we may assume that $w \in \{x_1, y_1, z_1\}$.

(i) If $w = x_1$, then by (4.3) we can take uy_1x_2u, wy_2z_1w and $y_3x_3y_4z_2y_3$.

(ii) If $w = y_1$, then we can take uy_2x_1u, wz_1z_2w and $y_3x_2y_4x_3y_3$.

(iii) If $w = z_1$, then we take uy_1x_1u, wy_2x_2w and $y_3x_3y_4z_2y_3$.

Thus if $G' = \mathbf{F_2}$, then either (Q2) or (Q4) holds.

CASE 4: (S6) holds for G', i.e. $k' = 3$ and $G' = \mathbf{F_3}$ in Fig. 3.2. So, $n' = 9$ and (Q5) holds.

CASE 5: (S7) holds for G', i.e. $k' \geq 3$, $|G'| = 3k'$, $\alpha(G') \leq k'$, and $\chi(\overline{G'}) > k'$. Since $|G'| = 3k'$, (4.1) must hold. Since $\chi(\overline{G'}) > k'$, G' contains an induced subgraph G_0 such that $\overline{G_0}$ is a vertex-$(k'+1)$-critical graph. By (4.1),

(4.4) for every $xy \in E(\overline{G_0})$, the sum of the degrees of x and y in $\overline{G_0}$ is at most $2k' + 1$.

The $(k'+1)$-critical graphs satisfying (4.4) were studied recently. If $k' \geq 5$, then by results in [8] and [15], $\overline{G_0} = K_{k'+1}$, which means $\alpha(G') \geq k' + 1$, a contradiction to the case. If $k' \leq 4$, then (Q5) holds.

5. Proof of Theorem 17, Case 6: $k' = 1$

In this section, we consider the case that (S1) holds for G', i.e. $k' = 1$ and G' is a forest with at most one isolated vertex. Since $k \geq 4$, there are strong edges $xz, x'z', x''z'' \in M$.

Call a vertex v *low* if $d_G(v) \leq 2k - 2$.

Case 6.1: $n > 2k+1$ and G' has at least two non-singleton components, say H_1 and H_2. Then $n' \geq 4$. For $i = 1, 2$, let P_i be a longest path in H_i, and let u_i and w_i be the ends of P_i. As $\mathcal{SO}(G) \geq 4k - 3$, at most two edges between W and $\{u_1, u_2, w_1, w_2\}$ are missing in G. So we may assume that at most one edge between $\{x, z\}$ and $\{u_1, u_2, w_1, w_2\}$ is missing in G. By symmetry, we assume that among these edges only xu_1 could be missing in G. Then the $\alpha' - 1$ strong edges of $M - xz$ and the cycles xu_2w_2 and zu_1w_1 form k disjoint cycles in G, a contradiction.

Case 6.2: $n > 2k + 1$ and G' has a unique non-singleton component H, and this H is not a star. Let $P = y_1, \ldots, y_t$ be a longest path in H. Since H is not a star, $t \geq 4$. Then y_1 is a leaf in G', and either $d_{G'}(y_2) = 2$ or y_2 is adjacent to a leaf $l \neq y_1$. Let $y_1' = y_2$ if

$d_H(y_2) = 2$ and $y'_1 = l$ otherwise. Similarly, either $d_{G'}(y_{t-1}) = 2$ or y_{t-1} is adjacent to a leaf $l' \neq y_t$. Let $y'_t = y_{t-1}$ if $d_H(y_{t-1}) = 2$ and $y'_t = l'$ otherwise. Since $y_1y'_t, y'_1y_t \notin E(G)$ and $G \in \mathcal{DO}_k$,

(5.1) *the number of missing edges between* $\{y_1, y'_1, y_t, y'_t\}$ *and* W *in* G *is at most* $q + r$*, where* $q = |\{y'_1, y'_t\} \cap \{y_2, y_{t-1}\}|$ *and* r *is the number of low vertices in* $\{y_1, y'_1, y_t, y'_t\}$.

Since $q \leq 2$, $r \leq 2$ and $|M| \geq 3$, for some edge $ab \in M$ at most one edge between $\{a, b\}$ and $\{y_1, y'_1, y_t, y'_t\}$ is missing in G. So we get a contradiction as at the end of Case 6.1.

Case 6.3: $n > 2k + 1$ and the unique non-singleton component H of G' is a star. Let x be the center of this star. Then $J = V(G') - x$ is a big set and $|J| = n' - 1 \geq 3$. So we have Case 1.

Case 6.4: $n = 2k + 1$. Then (Q7) holds.

6. Proof of Theorem 17 : Case 7: $G' \subseteq \mathbf{Y_{k',c,2k'-c}}$ and $k' > 2$

In this section we consider the case that (S8) holds for G', i.e. $n' = 3k'$ and $G' \subseteq \mathbf{Y_{k',c,2k'-c}}$ for $k' \geq 3$ and some odd $1 \leq c \leq k'$. If $k' \leq 3$, then (Q5) holds. So below in this section we assume

(6.1) $$k' \geq 4.$$

We view $V(G')$ in the form $V(G') = X \cup Z \cup Y$, where $|X| = c$, $|Z| = 2k' - c$, $|Y| = k'$, Y is independent and there are no edges between X and Z. First, we digress a bit:

Lemma 20. *Let* $t \geq 2$ *and* $\epsilon \in \{0, 1\}$*. Let* H *be a graph with* $V(H) = R \cup Q$ *such that* $|R| = 2t + \epsilon$, $|Q| = 3t - |R| = t - \epsilon$*, and let* $y_0 \in Q$*. If*

(1) each $u \in R$ *has at most one nonneighbor in* H *and*

(2) each $y \in Q - y_0$ *has at most* $1 + \epsilon$ *nonneighbors in* R *and*

(3) y_0 *has at most* 2 *nonneighbors in* R *and has only* $1 + \epsilon$ *nonneighbors if* $t = 2$.

then H *contains* t *vertex-disjoint triangles.*

Proof. Using induction, note the lemma holds for $t = 2$. If $t \geq 3$ then H has a triangle $T = y_0z_1z_2y_0$ with $z_1, z_2 \in R$. By induction $H' := H - T$ has $t - 1$ disjoint triangles. $\square$

Since $n' = 3k'$, we will often use (4.1). Since each $y \in Y$ has $k' - 1$ nonneighbors in Y, (4.1) yields

(6.2) $$|\overline{N}(y) - Y| + |\overline{N}(y') - Y| \leq 3 \quad \text{for all } y, y' \in Y.$$

By (6.2),

(6.3) $$\text{there is } y_0 \in Y \text{ such that } |\overline{N}(y)| \leq 1 \text{ for every } y \in Y - y_0.$$

Since each $x \in X$ has $2k' - c$ nonneighbors in Z and each $z \in Z$ has c nonneighbors in X, by (4.1) we may assume that

(6.4) $$|\overline{N}(x) \cap W| \leq 1 \text{ and } \overline{N}(x) \cap \overline{W} = Z \text{ for each } x \in X,$$

and

(6.5) $$|\overline{N}(z) - X| \leq 1 \text{ for each } z \in Z, \text{ and if } c = k' \text{ then } G[Z] = K_c.$$

Lemma 21. *Let $G' \subseteq \mathbf{Y_{k',c,2k'-c}}$ for $k' \geq 4$ and an odd $c \leq k'$. Suppose there are $w \in D'$ and $u \in W$ such that F has an M-alternating u,w-path P*

(A) If $w \in Y \cup Z$, then u has no neighbor in $Y - w$ or no neighbor in X.

(B) If $w \in X$, then u has no neighbor in Y or no neighbor in Z.

Proof. Let M' be the matching obtained from M by switching edges on P. Then $W(M') = W(M) - w + u$. Set $t = (2k' - c - 1)/2$. Since $1 \leq c \leq k'$ and is odd, by (6.1),

$$|Z| = 2k' - c \geq 5 \text{ and } k' - 1 \geq t \geq 2. \tag{6.6}$$

Arguing by contradiction, we assume the lemma fails and construct k disjoint cycles.

Case 1: $w \in Y \cup Z$. Since (A) does not hold, u has a neighbors $x \in X$ and $y \in Y - w$. Pick $y \in N(u) \cap Y - w$ with $s(y)$ minimum. Then for y_0 defined in (6.3), we have

$$\text{if } y_0 \in Y - w - y, \text{ then } y_0 u \notin E(G), \text{ and so by (6.2), } |\overline{N}(y_0) \cap Z| \leq 2. \tag{6.7}$$

By (6.4), $T := uxyu \subseteq G$. Set $\epsilon := 0$ if $w \in Z$; else $\epsilon := 1$. Partition $Y - y - w$ as $\{Q, \overline{Q}\}$ so that $|Q| = t - \epsilon$, $|\overline{Q}| = \frac{c-1}{2}$, and $y_0 \in \overline{Q} \cup \{w, y\}$ if $c > 1$. So $t \geq 3$, if $y_0 \in Q$. Regardless, by (6.3), (6.5) and (6.7), Q and $R := Z - w$ satisfy the conditions of Lemma 20. Thus $Q \cup R$ contains t disjoint triangles. By (6.4), $(X - x) \cup \overline{Q}$ contains $\frac{c-1}{2}$ disjoint triangles. Counting these $k' - 1$ triangles, T, and $k - k'$ strong edges of M' gives k disjoint cycles.

Case 2: $w \in X$. Since (B) fails, there are $z \in N(u) \cap Z$ and $y \in N(u) \cap Y$. Our first goal is to show there is an edge with ends in $N(u) \cap Y$ and $N(u) \cap Z$. If $N(u) \cap N(z) \neq \emptyset$ then we are done. Else, by (6.5), $N(z) \cap Y = Y - y = \overline{N}(u) \cap Y$. Let $y' \in Y - y$. By (6.2) applied to y and y', $|\overline{N}(y) \cap Z| \leq 2$. By (4.1) applied to u and y', $|\overline{N}(u) \cap Z| \leq 2$. By (6.6), $|Z| \geq 5$, so there is $z' \in Z \cap N(u) \cap N(y)$, and we are done.

Pick $xy \in E$ with $y \in N(u) \cap Y$ and $z \in N(u) \cap Z$ so that $s(y)$ is minimum. Then for y_0 defined in (6.3), using (6.2),

$$\text{if } y_0 \in Y - y \text{ then } |\overline{N}(y_0) \cap (Z - z)| \leq 2, \tag{6.8}$$

since $y_0 u \notin E(G)$ or $y_0 z \notin E(G)$.

Partition $Y - y$ as $\{Q, \overline{Q}\}$ so that $|Q| = t$, $|\overline{Q}| = \frac{c-1}{2}$, and $y_0 \in \overline{Q} + y$ if $c > 1$. So $t \geq 3$, if $y_0 \in Q$. Regardless, by (6.3), (6.5) and (6.8), Q and $R := Z - z$ satisfy the conditions of Lemma 20. Thus $Q \cup R$ contains t disjoint triangles. By (6.4), $(X - w) \cup \overline{Q}$ contains $\frac{c-1}{2}$ disjoint triangles. Counting these $k' - 1$ triangles, T, and $k - k'$ strong edges of M' gives k disjoint cycles. $\square$

Lemma 22. *Let $G' \subseteq \mathbf{Y_{k',c,2k'-c}}$ for $k' \geq 4$ and an odd $c \leq k'$. Then $|D \cap W| \leq 2$.*

Proof. Suppose $u \in D \cap W$. Then there is a matching M' and vertex $w_u \in V(G')$ such that $W(M') = W(M) + w_u - u$ and there is an M, M'-alternating path from u to w_u. By Lemma 21, u has no neighbors in $Y - w_u$ or in X or in Z.

By degree condition (4.1), there is at most one $u \in D \cap W$ with no neighbor in X or no neighbor in Z: otherwise for any $x \in X$ and $z \in Z$ we have the contradiction

$$\|\{x, z\}, W\| \leq 4\alpha' - 2 \text{ and so } s(x) + s(z) \leq 4k' - 2 + 4\alpha' - 2 \leq 4k - 4.$$

Similarly, there is at most one $u \in D \cap W$ with at most one neighbor in Y: otherwise, as $k' \geq 4$, there are two $y, y' \in Y$ with

$$\|\{y, y'\}, W\| \leq 4\alpha' - 4 \text{ and so } s(y) + s(y') \leq 4k' + 4\alpha' - 4 \leq 4k - 4.$$

Thus $|D \cap W| \leq 2$. $\square$

Lemma 22 yields that $|W| \leq 2 + |A| + |C|$. Thus (Q8) holds.

7. Proof of Theorem 17 : Case 8: $G' \subseteq \mathbf{Y_{k'-1,1,2k'}}$ and $k' > 2$

In this section we consider the case that (S9) holds for G', i.e. $n' = 3k'$ and $G' \subseteq \mathbf{Y_{k'-1,1,2k'}}$ for $k' \geq 3$. We view $V(G') = \{x\} \cup Z \cup Y$, where $|Z| = 2k'$, $|Y| = k' - 1$, Y is independent and there are no edges between x and Z. If $k' \leq 3$, then (Q5) holds. So as in Section 6, we assume (6.1).

Since $n' = 3k'$, we will often use (4.1). Since each $y \in Y$ has $k' - 2$ nonneighbors in Y, (4.1) yields

(7.1) $$|\overline{N}(y) - Y| + |\overline{N}(y') - Y| \leq 5 \quad \text{for all } y, y' \in Y.$$

This in turn yields:

(7.2) at most one $y \in Y$ has at least three nonneighbors in $V(G) - Y$; call it y_0, if exists.

Since x is not adjacent to any of the $2k'$ vertices in Z, by (4.1)

(7.3) $$N(x) = V(G) - Z - x \text{ and } N(z) = V(G) - x - z \text{ for each } z \in Z.$$

If x has a strong neighbor v_0 with the M-mate u_0, then we construct k disjoint cycles in G as follows. First, take the α' strong edges in $M - v_0u_0 + v_0x$. By (7.3), $G[Z] = K_{2k'}$ and each $y \in Y + u_0$ is adjacent to all of Z. So, we take k' 3-cycles each of which contains one vertex in $Y + u_0$ and two vertices in Z. This contradiction shows that $x \in D'$.

Suppose (Q9) does not hold. Since $x \in D'$, $d(x) = k + \alpha' - 1$ and M can play the role of M^* in the definition of (Q9), this means F has a maximum matching M' such that

(7.4) $$\text{there are } u_1, u_2 \in V(G) - V(M') - Z \text{ with } u_1u_2 \in E(G).$$

Similarly to the proof of Lemma 19, for $i = 1, 2$ the symmetric difference $M \triangle M'$ contains a path P_i of an even length an end of which is u_i. Since the other end w_i of P_i is not covered by M, $w_i \in V(G') \cap D$. Also by definition, none of the vertices in G' is an internal vertex in P_i. In particular, $x \notin V(P_i)$. Let M'' be the maximum matching in F such that $M \triangle M'' = P_1 \cup P_2$. Then $V(G) - V(M'') = V(G') - \{w_1, w_2\} \cup \{u_1, u_2\}$. If $|\{w_1, w_2\} \cap Z| = \ell_Z$ and $|\{w_1, w_2\} \cap Y| = \ell_Y$, then we can renumber the vertices in $Z - \{w_1, w_2\}$ and $Y - \{w_1, w_2\}$ as $z_1, \ldots, z_{2k'-\ell_Z}$, $y_1, \ldots, y_{k'-1-\ell_Y}$ and construct k disjoint cycles in G as follows. Take the $k - k'$ strong edges in M'', then take the cycle xu_1u_2x and for $j = 1, \ldots, k' - 1 - \ell_Y$ take the cycle (y_j, z_{2j-1}, z_{2j}). Finally, if $\ell_Y \geq 1$, then $|Z - \{z_1, \ldots, z_{2(k'-1-\ell_Y)}, w_1, w_2\}| = 3\ell_Y$, then we simply take ℓ_Y triangles in the remaining complete graph $G[Z - \{z_1, \ldots, z_{2(k'-1-\ell_Y)}, w_1, w_2\}]$. Hence (Q9) holds.

8. Proof of Theorem 17 : Case 9: (S2) holds for G'

Notation. WE NEED TO DECIDE WHERE THIS GOES. PROBABLY BEFORE THE THEOREM 13.

CHECK STATEMENT OF (Y4) AND (Y5) AND THEN CHANGE THEOREM 13.

WE USE $W = W(M)$ NOT $V(M)$. WHEN WE DEFINE $W(M)$ we should say, "Let $W = W(M)$ be the set of vertices of G that are saturated by M." I RECOMMEND $\overline{W}$ NOT V'. SOMETIMES WE ALSO USE $V(G')$.

(Q7) AND (Q10) ARE NOT USED.

In this section we consider the case that (S2) holds for k' and G', i.e., $n' \geq 3k'$ and $k' = 2$ and G' satisfies one of (Y1)–(Y5) from Theorem 13. If $n' = 6$ then (Q5) holds, so assume $n' \geq 7$. As $k \geq 5$, $|M| = \alpha' = k - k' \geq 3$.

Define a vertex $v \in \overline{W}$ to be i-*acceptable* if $|N(v) \cap W| \geq 2\alpha' - i$, *acceptable* if it is 1-acceptable, and *good* if it is 0-acceptable. Let $u, v \in \overline{W}$ with $uv \notin E$. If i and j are minimum natural numbers such that u is i-acceptable and v is j-acceptable, then

$$i + j \leq d_{G'}(u) + d_{G'}(v) - 5. \tag{8.1}$$

Case 9.1: G' satisfies (Y1), i.e., $G' \subseteq \mathbf{S_3}$. As $n' \geq 7$ and $G' \in \mathcal{DO}'_k$, $G' \in \{\mathbf{S_3}, \mathbf{S_3} - xz_1\}$. Regardless, by (8.1) x and z_1 are acceptable and the other vertices are good, so there is $ab \in M$ with $ax \in E$. Thus G has k disjoint cycles, $axya$, bz_4z_5b, $z_1z_2z_3z_1$ and $|M - ab|$ strong edges, contradicting $G \in \mathcal{BO}_k$.

Case 9.2: G' satisfies (Y2), i.e., $G' \subseteq \mathbf{S_4}$. As $n' \geq 7$ and $G' \in \mathcal{DO}'_k$, $G' = \mathbf{S_4}$. By (8.1), all vertices except x are good, and x is 2-acceptable. As $|M| \geq 3$, there is an edge $ab \in M$ with $ax \in E$. Again, G has k disjoint cycles, $axya$, bz_1z_5b, $z_4z_2z_3z_4$, and $|M - ab|$ strong edges, contradicting $G \in \mathcal{BO}_k$.

Case 9.3: G' satisfies (Y3), i.e., $G' = \mathbf{S_5}$. By (8.1), all vertices are acceptable. As $|M| \geq 3$, there is an edge $ab \in M$ with $|(N(a) \cup N(b)) \cap \{z_1, z_2, x, y\}| \geq 7$. Choose notation so that at worst $bz_1 \notin E$ or $bx \notin E$. Then az_1xa, bz_2yb, $z_3z_4z_5z_3$ and $|M - ab|$ strong edges yield k disjoint cycles, contradicting $G \in \mathcal{BO}_k$.

Case 9.4: G' satisfies (Y4), i.e., $G' \in \{H, \mathrm{sd}(H, e, x), \mathrm{sd}(H, e, xy)\}$, where $\mathbf{W_{|H|-1}} \subseteq H \subseteq \mathbf{W^+_{|H|-1}}$. Set $t = |H| - 1$. Let H have center v_0 and rim $v_1 \dots v_t v_1$, and let $\mathbf{W'_t} = \mathbf{W_t} \cup \mathbf{K}^*(\{v_0, v_1\})$ be the result of adding a parallel edge. Since G' is simple, we may assume $H \in \{\mathbf{W_t}, \mathbf{W_t}'\}$. If $G' \neq H$ then let $e = v_1w$ be the subdivided edge. As $n' \geq 7$, $t \geq 4$.

Case 9.4.1: $t = 4$. Then $G' = \mathrm{sd}(H, v_1w, xy)$. By (8.1), v_2, v_3, v_4, x, y are all good, and v_1 is acceptable (even if v_0v_1 is strong). Thus there is an edge $ab \in M$ with $av_1 \in E$. Then G contains k disjoint cycles av_4v_1a, $bxyb$, $v_0v_2v_3v_0$ and $\alpha' - 1$ strong edges, contradicting $G \in \mathcal{BO}_k$.

Case 9.4.2: $t = 5$. The subdividing vertex x exists. By (8.1), the subdividing vertices and v_3, v_4, v_5 are all good, v_2 is acceptable, and v_1 is 2-acceptable. As $|M| \geq 3$, there is an edge $ab \in M$ with $av_1, bv_2 \in E$. Then there are k disjoint cycles $v_0v_4v_5v_0$, av_1xa, bv_2v_2b, and $|M - ab|$ strong edges, contradicting $G \in \mathcal{BO}_k$.

Case 9.4.3: $t \geq 6$. By (8.1), the rim vertices v_3, v_4, v_5, v_6 are all acceptable. As $|M| \geq 3$, there is an edge $ab \in M$ such that av_3v_4a and bv_5v_6b are cycles. Let C be the smallest cycle containing v_0, v_1, v_2 (and any subdividing vertices). Then there are k disjoint cycles C, av_3v_4a, bv_5v_6b and $\alpha' - 1$ strong edges, contradicting $G \in \mathcal{BO}_k$.

Case 9.5: G' satisfies (Y5), i.e., $G' \in \{H, \mathrm{sd}(H, e, x), \mathrm{sd}(H, e, xy)\}$, where

$$\mathbf{K_{3,|H|-3}}(Y, Z_t) - e' \subseteq H \subseteq \mathbf{K^+_{3,|H|-3}} \text{ and } Y = \{y_1, y_2, y_3\}.$$

As $n' \geq 7$, $t \geq 2$. If $\alpha(G') \geq n' - 2k' + 1$ then (Q3) holds by Case 1. So assume the subdividing vertex x exists in G'.

Case 9.5.1: $e = y_h y_i$, where $\{h, i, j\} = [3]$. Since $\alpha(G') \leq n' - 2k'$ and $Z + x$ is independent, e is subdivided twice. As $d_{G'}(x) = 2$, every vertex of Z is adjacent to every vertex of Y (and no other vertex of G'). Thus $G' = \mathrm{sd}(H, e, xy)$ and the vertices of $Z + x + y$ are all good.

Suppose $t = 2$. As $y_j x \notin E$, y_j has a neighbor, say y_i, in Y. Since $d_{y_h} \leq 5$ and $y_h y \notin E$, (8.1) implies y_h is 2-acceptable. As $|M| \geq 3$, there is an edge $ab \in M$ with $av_h \in E$. Thus there are k disjoint cycles $ay_h z_1 a$, $bxyb$, $z_2 y_i y_j z_2$, and $\alpha' - 1$ other strong edges, contradicting $G \in \mathcal{BO}_k$.

Suppose $t = 3$. Then $d_{G'}(y_j) \leq 5$. By (8.1), y_j is 2-acceptable. As $|M| \geq 3$, there is an edge $ab \in M$ with $av_j \in E$. Thus there are k disjoint cycles $av_j z_1 a$, $bxyb$, $z_2 y_h z_3 y_i z_2$, and $\alpha' - 1$ strong edges, contradicting $G \in \mathcal{BO}_k$.

Otherwise $t \geq 4$. Then there are k disjoint cycles $axya$, $bz_1 y_1 z_2 b$, $z_3 y_2 z_4 y_3 z_3$, and $\alpha' - 1$ other strong edges, contradicting $G \in \mathcal{BO}_k$.

Case 9.5.2: $e \in E(Y, Z_t)$. Now H is simple. Say $e = y_1 z_1$ and $e' = y'z'$. If $e' \notin E(H)$ then $y' \neq y_1$. By degree conditions $xz' \in E$, so $z' = z_1$. As $xz_i, z_1 z_i \notin E$ for $i \geq 2$, (8.1) implies all vertices of $Z - z_1$ and all subdividing vertices are good, z_1 is acceptable, and z_1 is good if $e' \notin H$.

Case 9.5.2.1: $t = 2$. Since $n' \geq 7$, $G' = \mathrm{sd}(H, z_1 y_1, xy)$. As $\alpha(G') \leq n' - 2k'$, $Y + x$ is not independent. So there is an edge $y_h y_i$, where $[3] = \{h, i, j\}$. By (8.1), all of x, y, z_1, z_2 are good, and all of y_1, y_2, y_3 are acceptable. So there is an edge $ab \in M$ with $ay_j \in E$. If $j = 1$ then there are k disjoint cycles $ay_1 z_2 a$, $bxyb$, $z_1 y_2 y_3 z_1$, and $\alpha' - 1$ strong edges; else $j \neq 1$ and there are k disjoint cycles $ay_j z_1 a$, $bxyb$, $z_2 y_h y_i z_2$, and $\alpha' - 1$ strong edges. Anyway this contradicts $G \in \mathcal{BO}_k$.

Case 9.5.2.2: $t \geq 4$. Let $ab \in M$ with $a \in N(z_1)$. If $t \geq 5$ then there are k disjoint cycles, $az_1 xa, bz_2 y_1 z_3 b, z_4 y_2 z_5 y_3 z_4$, and $\alpha' - 1$ strong edges, contradicting $G \in \mathcal{BO}_k$. Else $t = 4$. Since $d_{G'}(y_2) \leq 6$ and $xy_2 \notin E$, (8.1) implies y_2 is 3-acceptable. As z_1 is acceptable and $|M| \geq 3$, there is an edge $ab \in M$ with $az_1, by_2 \in E$. As x and z_2 are good, this yields k disjoint cycles $az_1 xa$, $by_1 z_2 b$, $z_3 y_1 z_4 y_3 z_2$, and $\alpha' - 1$ strong edges, contradicting $G \in \mathcal{BO}_k$.

Case 9.5.2.3: $t = 3$ and $z_1 y_1$ is subdivided twice. Then x and y are both good. Since $d_{G'}(y_1) \leq 5$ and $xy_1 \notin E$, y_1 is 2-acceptable. As z_1 is acceptable, there is an edge $ab \in M$ with $ax, by_1 \in E$. Thus there are k disjoint cycles $az_1 xa$, $by_1 yb$, $y_2 z_2 y_3 z_3 y_2$, and $\alpha' - 1$ strong edges, contradicting $G \in \mathcal{BO}_k$.

Case 9.5.2.4: $t = 3$ and $z_1 y_1$ is subdivided once. Suppose there is an edge $y_i y_j \in E$, where $[3] = \{y_1, y_2, y_3\}$. Then $d_{G'}(y_h) \leq 5$ and either $y_h x \notin E$ or $y_h z_1 \notin E$. By (8.1), y_h is 3-acceptable. As $|M| \geq 3$, there is an edge $ab \in M$ with $az_1, by_h \in M$. Thus there are k disjoint cycles $az_1 xa$, $by_h z_2 b$, $y_i z_3 y_j z_3$, and $\alpha' - 1$ strong edges, contradicting $G \in \mathcal{BO}_k$. So assume $\|G[Y]\| = 0$.

If $|F| = 2\alpha'$ then (Q2b) holds. Else there are edges $ab, a'b' \in M$ and a vertex $u \in \overline{W}$ with $au \in E(F)$. All vertices of G' are good except one of y_1, z_1 might only be acceptable. Choose notation so that $\{b, a', b'\} = \{c_1, c_2, c_3\}$ and $|N(c_1) \cap \overline{W}| \geq 6$ and $|N(c_2) \cap \overline{W}|, |N(c_3) \cap \overline{W}| \geq 7$. By inspection $G' - u$ contains a perfect matching $\{e_1, e_2, e_3\}$ with $e_1 \subseteq N(c_1)$.

Thus G contains k disjoint cycles, $c_1e_1c_1$, $c_2e_2c_2$, $c_3e_3c_3$, aua and $\alpha' - 2$ other strong edges, contradicting $G \in \mathcal{BO}_k$. □

9. Proof of Theorem 18

To be completed. We define our algorithm in steps.

Step 1. Find F (in $O(n^2)$ operations) and a maximum matching M (in $O(n^3)$ operations). Let $\alpha' := \alpha'(F) = |M|$ and $n' = n - 2\alpha'$. If $n' < 3(k - \alpha')$, then G has no k disjoint cycles, otherwise go to Step 2.

Step 2. Construct a GE-decomposition (A, C, D) of $V(F)$ as follows: find the size $\alpha'(F - v)$ of a maximum matching in $F - v$ for all $v \in V(F)$ (in $O(n^4)$ operations). Then $D = \{v \in V(F) : \nu(F - v) = \nu(F)\}$, $A = N(F) - F$ and $C = V(F) - D - A$.

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