Math 412 **HW7**

Due Thursday, March 21, 2024

Solve four of the next five problems.

1. A *Tutte set* in a graph G is a subset S of vertices of G with $o(G - S) - |S| > 0$ (it does not exist if G has a perfect matching). Show that in a connected graph G of even order that does not have a perfect matching, every vertex in a smallest Tutte set S has neighbors in at least three odd components of $G - S$. Conclude that every graph of even order that does not contain an induced subgraph $K_{1,3}$ has a perfect matching.

2. Prove that for every simple graph G with maximum degree at most 3 and at least 4 vertices, $\kappa'(G) = \kappa(G)$. Give an example of a simple 4-regular graph G with $\kappa'(G) > \kappa(G)$.

3. Prove that a simple 2-connected graph G with at least four vertices is 3-connected if and only if for every triple (x, y, z) of distinct vertices and any edge e not incident with y, G has an x, z-path through e that does not contain y.

4. Let G be a simple n-vertex triangle-free graph with $\delta(G) \geq 3$. Prove that if $n \leq 11$, then G is 3-edge-connected. Show that such a graph could be not 2-(vertex)-connected. (Hint: Mantel's Theorem may help.)

5. Prove that in each ear decomposition of a 4-regular n-vertex 2-connected graph G, the number of ears (counting the initial cycle) is $n + 1$.

Problems below review basic concepts and their ideas could be used in the tests.

WARMUP PROBLEMS: Section 4.1: # 1, 2, 3, 5, 7. Section 4.2: # 1, 2, 3, 4, 6. Do not write these up!

OTHER INTERESTING PROBLEMS: Section 4.1: # 10, 12, 14, 24, 31.
Section 4.2: # 8, 9, 11, 12, 20, 21, 23, 24, 26. Do not write these up!