Matchings in bipartite graphs, II

Lecture 14

Hall's Theorem

Theorem 3.2 (P. Hall): An X, Y-bigraph G has a matching covering X if and only if

$$|N(S)| \geq |S| \qquad \forall S \subseteq X. \tag{1}$$

Corollary 3.3 (Marriage Theorem) For each $k \geq 1$ every k-regular bipartite graph has a perfect matching.

Vertex covers

A vertex cover of a graph G is a set S of vertices in G such that each edge of G has at least one end in S.

The minimum cardinality of a vertex cover of G is denoted by $\beta(G)$.

Observation A: A set $S \subset V(G)$ is a vertex cover if and only if $V(G) - S$ is an independent set.
Observation B: For each n-vertex graph G, $\alpha(G) + \beta(G) = n$.

Observation C: For each graph G, $\alpha'(G) \leq \beta(G) \leq 2\alpha'(G)$.

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$$\alpha'(G) = \beta(G). \tag{2}$$

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Proof. Let $G = (X, Y; E)$ be a bipartite graph with parts X and Y. By Observation C, we need only to prove $\alpha'(G) \geq \beta(G)$.

Let Q be a vertex cover of G with $|Q| = \beta(G)$.

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Let Q be a vertex cover of G with $|Q| = \beta(G)$.

Claim: (i) $\forall A \subseteq Q \cap X, \quad |N(A) - Q \cap Y| \geq |A|$.
(ii) $\forall B \subseteq Q \cap Y, \quad |N(B) - Q \cap X| \geq |B|$.

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Proof of Claim (i). If for some $A \subseteq Q \cap X$ $|N(A) - Q \cap Y| < |A|$, then the set $(Q - A) \cup N(A)$ is a smaller vertex cover.
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The proof of (ii) is symmetric.

By the claim and Hall's Theorem, graph $G[(Q \cap X) \cup (Y - Q)]$ has a matching M_X covering $Q \cap X$ and graph $G[(Q \cap Y) \cup (X - Q)]$ has a matching M_Y covering $Q \cap Y$.

Since M_X and M_Y are disjoint,

$$\alpha'(G) \geq |M_X| + |M_Y| = |Q \cap X| + |Q \cap Y| = |Q| = \beta(G).$$

This proves Theorem 3.4.

An edge cover of a graph G is a set T of edges in G such that each vertex of G is an end of at least one edge in T.

Trivially, if G has isolated vertices, then it has no edge cover. If G has no isolated vertices, then $E(G)$ is an edge cover of G. The problem is to find an edge cover of the minimum cardinality.

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Theorem 3.5 (Gallai, 1959): For each n-vertex graph G with no isolated vertices, $\alpha'(G) + \beta'(G) = n$.

Proof of Theorem 3.5. Let G be an n-vertex graph G with no isolated vertices.

Part 1: We prove $\alpha'(G) + \beta'(G) \leq n$. Let M be a matching in G with $|M| = \alpha'(G)$. It does not cover exactly $n - 2\alpha'(G)$ vertices. Each of these vertices we can cover with a special edge. Thus

$$\beta'(G) \leq \alpha'(G) + (n - 2\alpha'(G)) = n - \alpha'(G),$$

as claimed.

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Let L be an edge cover of G with $|L| = \beta'(G)$. Consider the subgraph G_L of G spanned by the edges in L.

By the minimality of L, G_L does not contain cycles and paths of length 3. Thus G_L is a star forest.

Let k be the number of components in G_L. Then G_L has a matching M with k edges.
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as claimed. $\square$

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On the other hand, $|L| = n - k$.

Therefore,

$$\beta'(G) + \alpha'(G) \geq |L| + |M| \geq (n - k) + k = n,$$

as claimed. □

Corollary: For each bipartite graph G with no isolated vertices, $\alpha(G) = \beta'(G)$.