Matchings in general graphs, II

Lecture 18

Theorem 3.7 (Tutte, 1947): A graph G has a p.m. if and only if

$$o(G - S) \leq |S| \qquad \forall S \subseteq V(G). \tag{1}$$

Theorem 3.8 (Petersen, 1891): Every 3-regular graph with no cut-edges has a p.m.

Theorem 3.9 (Petersen, 1891): For every $k \geq 1$, every $2k$-regular graph has a 2-factor.

Proof. It is enough to prove the theorem for connected graphs. So, suppose G is a connected $2k$-regular graph with vertex set $V = \{v_1, \ldots, v_n\}$. Then G has an Eulerian circuit C. Let $e_1, \ldots, e_m$ be the (directed) edges of C.

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We construct an auxiliary bigraph H as follows. The parts of H are V and $V' = \{v'_1, \ldots, v'_n\}$.
For every e_j in C, if e_j leads from v_i to v_h, we add edge $v_i v'_h$ to $E(H)$.

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So by Marriage Theorem, H has a p.m. M.

The edges of M form a 2-factor in G. □

Theorem 3.10 (Berge-Tutte Formula, 1958): For every graph G,

$$|V(G)| - 2\alpha'(G) = \max_{S \subseteq V(G)} \{o(G - S) - |S|\}. \qquad (2)$$

Proof. We will prove $\geq$ and $\leq$.

Part $\geq$. Let M a matching in G of size $\alpha'(G)$. Given any $S \subseteq V(G)$, M does not cover a vertex in at least $o(G - S) - |S|$ odd components of $G - S$. This yields

$$|V(G)| - 2\alpha'(G) \geq o(G - S) - |S|.$$

This proves our part.

Part $\leq$.

Let $d = \max_{S \subseteq V(G)}\{o(G - S) - |S|\}$. Trying $S = \emptyset$ yields $d \geq 0$. Moreover, by Tutte's Theorem, if $d = 0$, then we are done. So suppose $d \geq 1$.

Fix $S_0 \subseteq V(G)$ such that $d = o(G - S_0) - |S_0|$.

Let H be obtained from a copy of G and a disjoint from it copy of K_d with vertex set D by adding all edges with one end in $V(G)$ and one end in D.

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Since the parity of $d = o(G - S_0) - |S_0|$ is the same as of $o(G - S_0) + |S_0|$, which in turn is the same as the parity of n,

$$|V(H)| = n + d \qquad \text{is even.}$$

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We now claim that

$$H \quad \text{has a p.m.} \tag{3}$$

Indeed, suppose H has no p.m. Then by Tutte's Theorem, there is $T \subseteq V(H)$ s.t.

$$o(H - T) - |T| \geq 1. \tag{4}$$

Since $d \geq 1$, H is connected. This and the fact that $|V(H)|$ is even imply that $T \neq \emptyset$.

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If there is $w \in D - T$, then $H - T$ is connected, and hence $o(H - T) \leq 1$. This contradicts (4). Thus $D \subset T$.
It follows that

$$o(G - (T - D)) = o(H - T) \geq |T| + 1 = |T - D| + d + 1.$$

In other words, $o(G - (T - D)) - |T - D| \geq d + 1$,
contradicting the definition of d. □

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4. Petersen's Theorems (Corollary 3.8 and Theorem 3.9).
5. Berge-Tutte Formula (Theorem 3.10).
6. Gale-Shapley Algorithm and its proof.