Matchings in general graphs, III

Lecture 19

Theorem 3.7 (Tutte, 1947): A graph G has a p.m. if and only if

$$o(G - S) \leq |S| \qquad \forall S \subseteq V(G). \tag{1}$$

Theorem 3.8 (Petersen, 1891): Every 3-regular graph with no cut-edges has a p.m.

Theorem 3.9 (Petersen, 1891): For every $k \geq 1$, every $2k$-regular graph has a 2-factor.

Corollary (Petersen, 1891): For every $k \geq 1$, the edges of every $2k$-regular graph partition into k 2-factors.

Theorem 3.10 (Berge-Tutte Formula, 1958): For every graph G,

$$|V(G)| - 2\alpha'(G) = \max_{S \subseteq V(G)}\{o(G - S) - |S|\}. \qquad (2)$$

Proof. We will prove $\geq$ and $\leq$.

Part $\geq$. Let M a matching in G of size $\alpha'(G)$. Given any $S \subseteq V(G)$, M does not cover a vertex in at least $o(G - S) - |S|$ odd components of $G - S$. This yields

$$|V(G)| - 2\alpha'(G) \geq o(G - S) - |S|.$$

This proves our part.

Part $\leq$.

Let $d = \max_{S \subseteq V(G)} \{o(G - S) - |S|\}$. Trying $S = \emptyset$ yields $d \geq 0$. Moreover, by Tutte's Theorem, if $d = 0$, then we are done. So suppose $d \geq 1$.

Fix $S_0 \subseteq V(G)$ such that $d = o(G - S_0) - |S_0|$.

Let H be obtained from a copy of G and a disjoint from it copy of K_d with vertex set D by adding all edges with one end in $V(G)$ and one end in D.

Part $\leq$.

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Since the parity of $d = o(G - S_0) - |S_0|$ is the same as of $o(G - S_0) + |S_0|$, which in turn is the same as the parity of n,

$$|V(H)| = n + d \qquad \text{is even.}$$

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$$|V(H)| = n + d \qquad \text{is even.}$$

We now claim that

$$H \quad \text{has a p.m.} \tag{3}$$

Indeed, suppose H has no p.m. Then by Tutte's Theorem, there is $T \subseteq V(H)$ s.t.

$$o(H - T) - |T| \geq 1. \tag{4}$$

Since $d \geq 1$, H is connected. This and the fact that $|V(H)|$ is even imply that $T \neq \emptyset$.

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If there is $w \in D - T$, then $H - T$ is connected, and hence $o(H - T) \leq 1$. This contradicts (4). Thus $D \subset T$.
It follows that

$$o(G - (T - D)) = o(H - T) \geq |T| + 1 = |T - D| + d + 1.$$

In other words, $o(G - (T - D)) - |T - D| \geq d + 1$, contradicting the definition of d. □

Corollary 3.11: If a graph G with an even number of vertices has no p.m., then there is an $S \subset V(G)$ s.t. $o(G - S) - |S| \geq 2$.

Main theorems in Chapter 3:

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4. Petersen's Theorems (Theorems 3.8 and 3.9).
5. Berge-Tutte Formula (Theorem 3.10).
6. Gale-Shapley Algorithm and its proof.

In many applications of Graph Theory one needs a measure of how vulnerable for a given connected graph is its connectedness, i.e. how difficult is to make a graph disconnected. For example, for large n, the graph K_n seems more "reliable" than the graph P_n.

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The connectivity of G, $\kappa(G)$, is the minimum k s.t. for some $S \subseteq V(G)$ with $|S| = k$, graph $G - S$ either is disconnected or has at most one vertex.

Note that with this definition, $\kappa(K_1) = 0$. (Also for each 1-vertex graph.)

Lemma 4.1: For every connected n-vertex graph G, $\kappa(G)$ is the minimum of $n-1$ and the size of a minimum separating set .

Proof. If an n-vertex graph G has a separating set S, then $G - S$ has at least two vertices. Hence in this case the connectivity is the minimum size of a separating set.

If our G has no separating sets, then each vertex is adjacent to each other vertex, and the connectivity is $n-1$. □

Connectivity of K_n and $K_{n,m}$.

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Connectivity of K_n and $K_{n,m}$.

A graph G is **k-connected** if $\kappa(G) \geq k$.

In particular, each **$(k+1)$-connected** graph is also **k-connected**.

A disconnecting set of edges in a graph G is a $T \subset E(G)$ s.t. $G - T$ is disconnected.

For a graph G with at least two vertices, the edge connectivity of G, $\kappa'(G)$, is the cardinality of a minimum disconnecting set. The edge connectivity of each 1-vertex graph is defined to be 0. In particular, $\kappa'(K_1) = 0$.

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For $S \subset V(G)$ we denote by $E(S, \overline{S})$ the set of edges of G connecting S with $\overline{S}$.

Observation: If T is a disconnecting set in G with $|T| = \kappa'(G)$, then T is an edge cut.
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A graph G is **k-edge-connected** if $\kappa'(G) \geq k$.