Connectivity

Lecture 20

In many applications of Graph Theory one needs a measure of how vulnerable for a given connected graph is its connectedness, i.e. how difficult is to make a graph disconnected. For example, for large n, the graph K_n seems more "reliable" than the graph P_n.

We will study the most popular measures.

A separating set (vertex cut) in a graph G is an $S \subset V(G)$ s.t. $G - S$ is disconnected.

Observe that K_n has no separating sets.

The connectivity of G, $\kappa(G)$, is the minimum k s.t. for some $S \subseteq V(G)$ with $|S| = k$, graph $G - S$ either is disconnected or has at most one vertex.

Note that with this definition, $\kappa(K_1) = 0$. (Also for each 1-vertex graph.)

Lemma 4.1: For every connected n-vertex graph G, $\kappa(G)$ is the minimum of $n-1$ and the size of a minimum separating set .

Proof. If an n-vertex graph G has a separating set S, then $G-S$ has at least two vertices. Hence in this case the connectivity is the minimum size of a separating set.

If our G has no separating sets, then each vertex is adjacent to each other vertex, and the connectivity is $n-1$. □

Connectivity of K_n and $K_{n,m}$.

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Connectivity of K_n and $K_{n,m}$.

A graph G is **k-connected** if $\kappa(G) \geq k$.

In particular, each **$(k+1)$-connected** graph is also **k-connected**.

A disconnecting set of edges in a graph G is a $T \subset E(G)$ s.t. $G - T$ is disconnected.

For a graph G with at least two vertices, the edge connectivity of G, $\kappa'(G)$, is the cardinality of a minimum disconnecting set. The edge connectivity of each 1-vertex graph is defined to be 0. In particular, $\kappa'(K_1) = 0$.

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An edge cut in a graph G is the set of edges of G connecting the vertices of some $S \subset V(G)$ with $\overline{S} = V(G) - S$.

For $S \subset V(G)$ we denote by $E(S, \overline{S})$ the set of edges of G connecting S with $\overline{S}$.

Observation: If T is a disconnecting set in G with $|T| = \kappa'(G)$, then T is an edge cut.
(Otherwise, T would not be minimum.)

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(Otherwise, T would not be minimum.)

A graph G is **k-edge-connected** if $\kappa'(G) \geq k$.

Theorem 4.2: For every graph G, $\kappa(G) \leq \kappa'(G) \leq \delta(G)$.

Proof. For 1-vertex graphs and disconnected graphs, the claim is trivial. Suppose $|V(G)| = n \geq 2$ and G is connected.

The inequality $\kappa'(G) \leq \delta(G)$ is easy: deleting from G the $\delta(G)$ edges incident to a vertex v of the minimum degree makes the remaining graph disconnected.

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Let us prove $\kappa(G) \leq \kappa'(G)$. By the observation above, there is $S \subseteq V(G)$ s.t. $\kappa'(G) = |E(S, \overline{S})|$. Let $s = |S|$.

Case 1: Each $x \in S$ is adjacent to each $y \in \overline{S}$. Then

$$\kappa'(G) = |E(S, \overline{S})| \geq s(n-s) \geq 1(n-1) = n-1.$$

But by Lemma 4.1, $\kappa(G) \leq n-1$, so we are done.

Case 2: There is $x \in S$ not adjacent to some $y \in \overline{S}$.

Let $R = (N(x) \cap \overline{S}) \cup \{z \in S - x : z$ has a neighbor $z' \in \overline{S}\}$.

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Claim 1: $G - R$ is disconnected.

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Claim 2: $|R| \leq |E(S, \overline{S})| = \kappa'(G)$.

Indeed, with each vertex $v \in N(x) \cap \overline{S}$ we can associate edge $xv \in E(S, \overline{S})$, and with each vertex $z \in S - x$ s.t. z has a neighbor $z' \in \overline{S}$ we can associate edge $zz' \in E(S, \overline{S})$.

These claims together prove the theorem. $\square$

The **differences** $\delta(G) - \kappa'(G)$ and $\kappa'(G) - \kappa(G)$ can be arbitrarily large. However, the following is true.

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Proof. In view of Theorem 4.2, it is enough to find for every 3-regular graph G with $|V(G)| \geq 4$ an edge cut with $\kappa(G)$ edges. For a 3-regular graph G, $0 \leq \kappa(G) \leq 3$.

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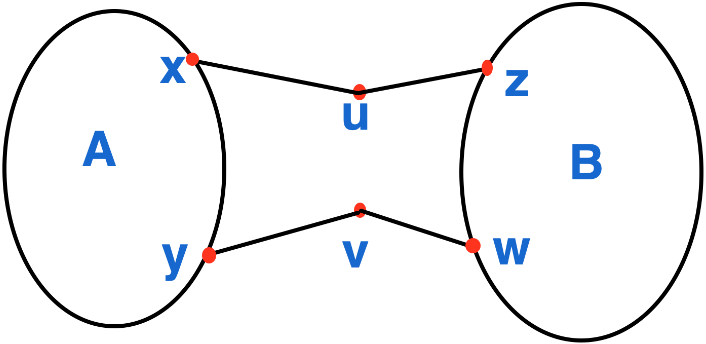
Case 1: $\kappa(G) = 1$. Then, since $|V(G)| \geq 4$, G has a cut vertex, say v. This means $V(G) = \{v\} \cup A \cup B$, s.t. there are no edges between A and B. Since $d(v) = 3$, either $|E_G(A, \{v\})| = 1$ or $|E_G(B, \{v\})| = 1$. By symmetry, we may assume $|E_G(A, \{v\})| = 1$ and u is the neighbor of v in A. Then uv is a cut edge in G, and hence $\kappa'(G) = 1$, as claimed.

Case 2: $\kappa(G) = 2$. Since $|V(G)| \geq 4$, G has a 2-vertex separating set, say $\{v, u\}$. This means $V(G) = \{v, u\} \cup A \cup B$, s.t. there are no edges between A and B.



Since $\kappa(G) = 2$, each of v, u has a neighbor in A and a neighbor in B, as in the picture. If $\kappa'(G) > 2$, then there is a third edge between A and $\{u, v\}$, say ux'.
Similarly, there is a third edge between B and $\{u, v\}$. Since already know all edges incident to u, this is an edge incident to v, say vw'.

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But then we know all edges incident to u or v, so we see that the edges uz and yv separate $A \cup \{u\}$ from $B \cup \{v\}$. This finishes Case 2.

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Case 3: $\kappa(G) = 3$. By Theorem 4.2, $\kappa'(G) \leq \delta(G) = 3$. So again the theorem holds. □

Two u, v-paths are internally disjoint, if they do not have common internal vertices.

Theorem 4.4 (Whitney, 1932). Let $|V(G)| \geq 3$. Then G is 2-connected if and only if for each $u, v \in V(G)$ graph G has internally disjoint u, v-paths.

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Proof. Let $n \geq 3$.
$(\Leftarrow)$ We prove the contrapositive. Suppose an n-vertex G is not 2-connected. Since $n \geq 3$, by Lemma 4.1 there is an $x \in V(G)$ such that $G - x$ is disconnected. This means there is a partition $V(G) = \{x\} \cup A \cup B$ with $A \neq \emptyset$ and $B \neq \emptyset$ such that no edge connects A with B.

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Let $a \in A$ and $b \in B$. Then each a, b-path in G contains x. Thus G has no internally disjoint a, b-paths.