Menger's Theorem and its variations

Lecture 23

Main version of Menger's Theorem

Let G be a graph or a digraph and $x, y \in V(G)$ with $xy \notin E(G)$. Then an x, y-cut is a set $S \subset V(G) - \{x, y\}$ such that $G - S$ has no x, y-paths.
Define $\kappa_G(x, y)$ be the minimum size of an x, y-cut in G.
Also, by $\lambda_G(x, y)$ denote the maximum number of internally disjoint x, y-paths in G.

Clearly, $\kappa_G(x, y) \geq \lambda_G(x, y)$.

Theorem 4.9 (Menger): Let G be a graph, $x, y \in V(G)$ and $xy \notin E(G)$. Then $\kappa_G(x, y) = \lambda_G(x, y)$.

Proof. Assume the theorem does not hold. Then there is a counterexample, i.e. a graph G and two vertices $x, y \in V(G)$ with $xy \notin E(G)$ such that

$$\kappa_G(x, y) > \lambda_G(x, y) \tag{1}$$

with the minimum $|V(G)|$. Let $n = |V(G)|$.

Proof setup

By the minimality of $|V(G)|$, for each H with $|V(H)| < n$

$$\kappa_H(u,v) = \lambda_H(u,v) \text{ for each } u,v \in V(H) \text{ with } uv \notin E(H). \quad (2)$$

Let $k = \kappa_G(x,y)$. If $k = 0$, then also $\lambda_G(x,y) = 0$, and the theorem holds. So assume $k \geq 1$.

Since $N(x)$ and $N(y)$ are x,y-cuts,

$$k \leq \min\{|N(x)|, |N(y)|\}. \quad (3)$$

In a series of claims below, we derive more and more

properties of G. Eventually, we will show that it does not exist.

Claims and conclusion

Claim 1. Every x, y-cut with k vertices is $N(x)$ or $N(y)$.

Claim 2. $V(G) = \{x, y\} \cup N(x) \cup N(y)$.

Claim 3. $N(x) \cap N(y) = \emptyset$.

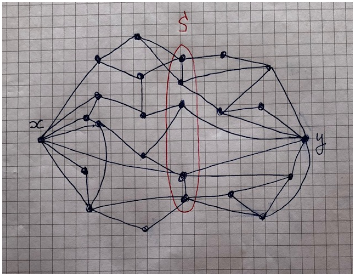
Claim 4. $|N(x)| = |N(y)| = k$.

Claim 5. Let H be the bipartite graph with parts $N(x)$ and $N(y)$ such that $u \in N(x)$ is adjacent to $v \in N(y)$ iff $uv \in E(G)$. Then H has a perfect matching.

Since a perfect matching in H corresponds to k internally disjoint x, y-paths in G, Claim 5 yields that $\lambda_G(x, y) \geq k$, a contradiction to (1).

Proof of Claim 1.

Suppose G has an x, y-cut S with k vertices distinct from $N(x)$ or $N(y)$.



Let G' be the component of $G - S$ containing x. Let G_x be obtained from $G - G'$ by adding the new vertex x' adjacent to all vertices of S. Graph G_y is defined symmetrically, but instead of $G - G'$ it uses $G[S \cup V(G')]$.

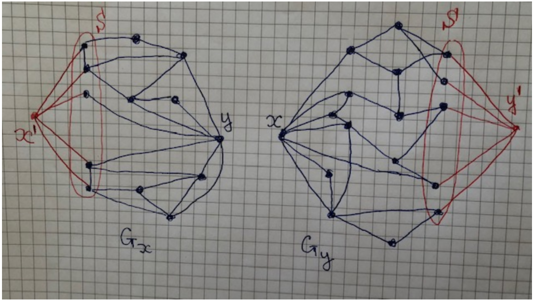
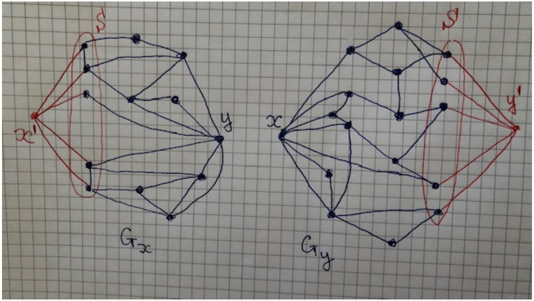


Figure: Graphs G_x and G_y.

Since S does not contain $N(x)$ or $N(y)$, each of G_x and G_y is smaller than G.



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Since S does not contain $N(x)$ or $N(y)$, each of G_x and G_y is smaller than G.

Any x', y-cut S' in G_x is also an x, y-cut in G. It follows that $\kappa_{G_x}(x', y) \geq k$. In view of S, it is exactly k. So by the minimality of G, $\lambda_{G_x}(x', y) = k$.

Similarly, $\lambda_{G_y}(x, y') = k$.

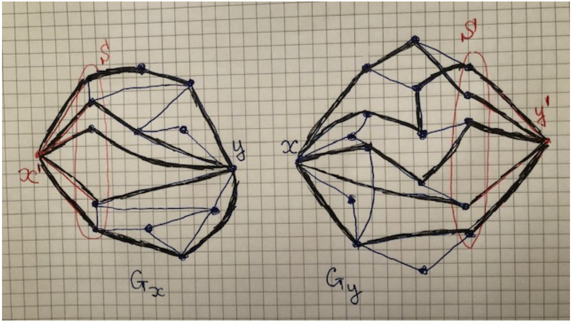

Figure: Paths in graphs G_x and G_y.

Let $P_1, \ldots, P_k$ be int.-disjoint x', y-paths in G_x and $Q_1, \ldots, Q_k$ be int.-disjoint x, y'-paths in G_y.

Then for every $1 \leq i \leq k$, $R_i = (Q_i - y') \cup (P_i - x')$ is an x, y-path in G. Also, all $R_1, \ldots, R_k$ are int.-disjoint, contradicting (1).

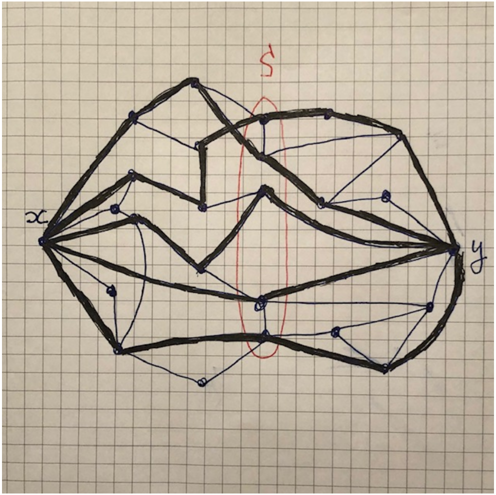

Figure: x, y-paths in G.

This proves Claim 1.

Proof of Claim 2: $V(G) = \{x, y\} \cup N(x) \cup N(y)$.

Suppose G has a vertex $z \in V(G) - (\{x, y\} \cup N(x) \cup N(y))$. By Claim 1, z does not belong to any x, y-cut of size k. This means that for the graph $G' = G - z$

$$\kappa_{G'}(x, y) = k.$$

By the minimality of G,

$$\lambda_{G'}(x, y) = \kappa_{G'}(x, y) = k.$$

So,

$$\lambda_G(x, y) \geq \lambda_{G'}(x, y) = k,$$

contradicting (1).

Proof of Claim 3: $N(x) \cap N(y) = \emptyset$.

Suppose G has a vertex $u \in N(x) \cap N(y)$.

Let $G' = G - u$. Then $\kappa_{G'}(x, y) \geq k - 1$.

Proof of Claim 3: $N(x) \cap N(y) = \emptyset$.
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Let $G' = G - u$. Then $\kappa_{G'}(x, y) \geq k - 1$.
By the minimality of G, $\lambda_{G'}(x, y) = \kappa_{G'}(x, y) \geq k - 1$. Let $P_1, \ldots, P_{k-1}$ be int.-disjoint x, y-paths in G'.
Adding to them path $P_k = x, u, y$ we obtain k int.-disjoint x, y-paths in G. This contradicts (1).

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Suppose $|N(x)| \geq k + 1$ and $v \in N(x)$. Let $G' = G - v$.

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Hence $\kappa_{G'}(x, y) = k$.

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Hence $\kappa_{G'}(x, y) = k$.
By the minimality of G, $\lambda_{G'}(x, y) = \kappa_{G'}(x, y) = k$. So,

$$\lambda_G(x, y) \geq \lambda_{G'}(x, y) = k,$$

contradicting (1).

Proof of Claim 5: The auxiliary bigraph H has a perfect matching.

Recall that H is the bipartite graph obtained from G by deleting x and y and all edges inside $N(x)$ and $N(y)$. Also recall that by Claim 4, $|N(x)| = |N(y)| = k$.

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But by the choice of A,

$$|S| = |N(x) - A| + |N_H(A)| = k - |A| + |N_H(A)| < k,$$

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This proves Theorem 4.9.

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In the former case, $|V(G - e)| \le |S| + 1 \le k - 1$.

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Thus $(G - e) - S$ is disconnected, i.e., it has a vertex partition into A and B such that no edges connect A with B in $(G - e) - S$.

Since $|S| < k = \kappa(G)$, S is not separating in G. Hence, one of x, y is in A and the other in B. Say $x \in A$ and $y \in B$.

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But then $|V(G)| = 2 + |S| \leq k$, contradicting the fact that $\kappa(G) \leq |V(G)| - 1$ for all G.

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Remark. The proof implies that if e is parallel to another edge, then $\kappa(G - e) = \kappa(G)$.

Global Menger's Theorem

The following theorem shows how k-connectedness refines itself.

Theorem 4.11 (Menger) : Suppose $n \geq k + 1$. Then an n-vertex graph G is k-connected if and only if $\lambda_G(x, y) \geq k$ for all distinct $x, y \in V(G)$.

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Proof. $(\Leftarrow)$ We prove the contrapositive. Suppose an n-vertex G is not k-connected. Since $n \geq k+1$, there is an $S \subseteq V(G)$ with $|S| \leq k-1$ such that $G - S$ is disconnected. This means there is a partition $V(G) = S \cup A \cup B$ with $A \neq \emptyset$ and $B \neq \emptyset$ such that no edge connects A with B.

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Let $a \in A$ and $b \in B$. Then each a, b-path in G contains a vertex of S. Since $|S| \leq k-1$, $\lambda_G(a, b) \leq k-1$, as claimed.

$(\Rightarrow)$ Let G be k-connected. Take any distinct $x, y \in V(G)$.
Case 1: $xy \notin E(G)$. Since G is k-connected, $\kappa_G(x, y) \geq k$. So by Theorem 4.9, $\lambda_G(x, y) \geq k$.

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Consider $G' = G - \{e_1, \ldots, e_s\}$. By Lemma 4.10, $\kappa(G') \geq k - s$. Also $xy \notin E(G')$.

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So, Case 1 applies to G', and hence $\lambda_{G'}(x, y) \geq k - s$. Together with the s edges $e_1, \ldots, e_s$ we get $(k - s) + s = k$ int.-disjoint x, y-paths, as claimed.

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Remark. Condition $n \geq k + 1$ is important here. Indeed, consider the graph G obtained from C_3 by replacing each edge with 1000 multiple edges. Then the connectivity of G is 2, but for any two vertices $x, y \in E(G)$, $\lambda_G(x, y) = 1001$.