Menger's Theorem and its variations, II

Lecture 24

Theorem 4.9 (Menger): Let G be a graph, $x, y \in V(G)$ and $xy \notin E(G)$. Then $\kappa_G(x, y) = \lambda_G(x, y)$.

Lemma 4.10: Deletion of an edge from a graph decreases connectivity by at most 1.

Global Menger's Theorem

The following theorem shows how k-connectedness refines itself.

Theorem 4.11 (Menger) : Suppose $n \geq k + 1$. Then an n-vertex graph G is k-connected if and only if $\lambda_G(x, y) \geq k$ for all distinct $x, y \in V(G)$.

Global Menger's Theorem

The following theorem shows how k-connectedness refines itself.

Theorem 4.11 (Menger) : Suppose $n \geq k+1$. Then an n-vertex graph G is k-connected if and only if $\lambda_G(x, y) \geq k$ for all distinct $x, y \in V(G)$.

Proof. ($\Leftarrow$) We prove the contrapositive. Suppose an n-vertex G is not k-connected. Since $n \geq k+1$, there is an $S \subseteq V(G)$ with $|S| \leq k-1$ such that $G - S$ is disconnected. This means there is a partition $V(G) = S \cup A \cup B$ with $A \neq \emptyset$ and $B \neq \emptyset$ such that no edge connects A with B.

Global Menger's Theorem

The following theorem shows how k-connectedness refines itself.

Theorem 4.11 (Menger) : Suppose $n \geq k + 1$. Then an n-vertex graph G is k-connected if and only if $\lambda_G(x, y) \geq k$ for all distinct $x, y \in V(G)$.

Proof. ($\Leftarrow$) We prove the contrapositive. Suppose an n-vertex G is not k-connected. Since $n \geq k + 1$, there is an $S \subseteq V(G)$ with $|S| \leq k - 1$ such that $G - S$ is disconnected. This means there is a partition $V(G) = S \cup A \cup B$ with $A \neq \emptyset$ and $B \neq \emptyset$ such that no edge connects A with B.

Let $a \in A$ and $b \in B$. Then each a, b-path in G contains a vertex of S. Since $|S| \leq k - 1$, $\lambda_G(a, b) \leq k - 1$, as claimed.

$(\Rightarrow)$ Let G be k-connected. Take any distinct $x, y \in V(G)$.
Case 1: $xy \notin E(G)$. Since G is k-connected, $\kappa_G(x, y) \geq k$. So by Theorem 4.9, $\lambda_G(x, y) \geq k$.

$(\Rightarrow)$ Let G be k-connected. Take any distinct $x, y \in V(G)$.

Case 1: $xy \notin E(G)$. Since G is k-connected, $\kappa_G(x, y) \geq k$. So by Theorem 4.9, $\lambda_G(x, y) \geq k$.

Case 2: G has exactly $s > 0$ edges connecting x with y. Let these edges be $e_1, \ldots, e_s$.

$(\Rightarrow)$ Let G be k-connected. Take any distinct $x, y \in V(G)$.
Case 1: $xy \notin E(G)$. Since G is k-connected, $\kappa_G(x, y) \geq k$. So by Theorem 4.9, $\lambda_G(x, y) \geq k$.

Case 2: G has exactly $s > 0$ edges connecting x with y. Let these edges be $e_1, \ldots, e_s$.
Consider $G' = G - \{e_1, \ldots, e_s\}$. By Lemma 4.10, $\kappa(G') \geq k - s$. Also $xy \notin E(G')$.

($\Rightarrow$) Let G be k-connected. Take any distinct $x, y \in V(G)$.
Case 1: $xy \notin E(G)$. Since G is k-connected, $\kappa_G(x, y) \geq k$. So by Theorem 4.9, $\lambda_G(x, y) \geq k$.

Case 2: G has exactly $s > 0$ edges connecting x with y. Let these edges be $e_1, \ldots, e_s$.
Consider $G' = G - \{e_1, \ldots, e_s\}$. By Lemma 4.10, $\kappa(G') \geq k - s$. Also $xy \notin E(G')$.

So, Case 1 applies to G', and hence $\lambda_{G'}(x, y) \geq k - s$. Together with the s edges $e_1, \ldots, e_s$ we get $(k - s) + s = k$ int.-disjoint x, y-paths, as claimed.

($\Rightarrow$) Let G be k-connected. Take any distinct $x, y \in V(G)$.
Case 1: $xy \notin E(G)$. Since G is k-connected, $\kappa_G(x, y) \geq k$. So by Theorem 4.9, $\lambda_G(x, y) \geq k$.

Case 2: G has exactly $s > 0$ edges connecting x with y. Let these edges be $e_1, \ldots, e_s$.
Consider $G' = G - \{e_1, \ldots, e_s\}$. By Lemma 4.10, $\kappa(G') \geq k - s$. Also $xy \notin E(G')$.

So, Case 1 applies to G', and hence $\lambda_{G'}(x, y) \geq k - s$.
Together with the s edges $e_1, \ldots, e_s$ we get $(k - s) + s = k$ int.-disjoint x, y-paths, as claimed.

Remark. Condition $n \geq k + 1$ is important here. Indeed, consider the graph G obtained from C_3 by replacing each edge with 1000 multiple edges. Then the connectivity of G is 2, but for any two vertices $x, y \in E(G)$, $\lambda_G(x, y) = 1001$.

Edge version of Menger's Theorem

Let G be a graph or a digraph and $x, y \in V(G)$. Then an x, y-edge-cut is a set $L \subset E(G)$ such that $G - L$ has no x, y-paths. Define $\kappa'_G(x, y)$ be the minimum size of an x, y-edge-cut in G.

Also, by $\lambda'_G(x, y)$ denote the maximum number of edge-disjoint x, y-paths in G.

Edge version of Menger's Theorem

Let G be a graph or a digraph and $x, y \in V(G)$. Then an x, y-edge-cut is a set $L \subset E(G)$ such that $G - L$ has no x, y-paths. Define $\kappa'_G(x, y)$ be the minimum size of an x, y-edge-cut in G.
Also, by $\lambda'_G(x, y)$ denote the maximum number of edge-disjoint x, y-paths in G.

Note that in this case there is no restriction $xy \notin E(G)$.

Edge version of Menger's Theorem

Let G be a graph or a digraph and $x, y \in V(G)$. Then an x, y-edge-cut is a set $L \subset E(G)$ such that $G - L$ has no x, y-paths. Define $\kappa'_G(x, y)$ be the minimum size of an x, y-edge-cut in G.
Also, by $\lambda'_G(x, y)$ denote the maximum number of edge-disjoint x, y-paths in G.

Note that in this case there is no restriction $xy \notin E(G)$.

Again, clearly, $\kappa'_G(x, y) \geq \lambda'_G(x, y)$.

Theorem 4.12 (Menger): Let G be a graph, $x, y \in V(G)$. Then $\kappa'_G(x, y) = \lambda'_G(x, y)$.

Edge version of Menger's Theorem

Let G be a graph or a digraph and $x, y \in V(G)$. Then an x, y-edge-cut is a set $L \subset E(G)$ such that $G - L$ has no x, y-paths. Define $\kappa'_G(x, y)$ be the minimum size of an x, y-edge-cut in G.
Also, by $\lambda'_G(x, y)$ denote the maximum number of edge-disjoint x, y-paths in G.

Note that in this case there is no restriction $xy \notin E(G)$.

Again, clearly, $\kappa'_G(x, y) \geq \lambda'_G(x, y)$.

Theorem 4.12 (Menger): Let G be a graph, $x, y \in V(G)$. Then $\kappa'_G(x, y) = \lambda'_G(x, y)$.

We will prove the theorem later using flows in networks.

Digraph versions of Menger's Theorem

Let G be a digraph. Then both versions of local Menger's Theorems, Theorems 4.9 and 4.12 also hold for digraphs:

Theorem 4.13 (Menger): Let G be a digraph, $x, y \in V(G)$ and $xy \notin E(G)$. Then $\kappa_G(x, y) = \lambda_G(x, y)$.

Theorem 4.14 (Menger): Let G be a digraph, $x, y \in V(G)$. Then $\kappa'_G(x, y) = \lambda'_G(x, y)$.

Digraph versions of Menger's Theorem

Let G be a digraph. Then both versions of local Menger's Theorems, Theorems 4.9 and 4.12 also hold for digraphs:

Theorem 4.13 (Menger): Let G be a digraph, $x, y \in V(G)$ and $xy \notin E(G)$. Then $\kappa_G(x, y) = \lambda_G(x, y)$.

Theorem 4.14 (Menger): Let G be a digraph, $x, y \in V(G)$. Then $\kappa'_G(x, y) = \lambda'_G(x, y)$.

We will prove the theorems later using flows in networks.

Fans and Fan Lemma

Let G be a graph, $U \subset V(G)$ and $x \in V(G) - U$. Then an x, U-fan of size k in G is a set of k paths from x to U such that any two of them share only x.

Fans and Fan Lemma

Let G be a graph, $U \subset V(G)$ and $x \in V(G) - U$. Then an x, U-fan of size k in G is a set of k paths from x to U such that any two of them share only x.

Theorem 4.15 (Fan Lemma): Let $n \geq k + 1$. An n-vertex graph G is k-connected if and only if for every choice $U \subset V(G)$ with $|U| \geq k$ and $x \in V(G) - U$, G has an x, U-fan of size k.

Fans and Fan Lemma

Let G be a graph, $U \subset V(G)$ and $x \in V(G) - U$. Then an x, U-fan of size k in G is a set of k paths from x to U such that any two of them share only x.

Theorem 4.15 (Fan Lemma): Let $n \geq k + 1$. An n-vertex graph G is k-connected if and only if for every choice $U \subset V(G)$ with $|U| \geq k$ and $x \in V(G) - U$, G has an x, U-fan of size k.

Proof. $(\Rightarrow)$ Suppose G is k-connected, and $U \subset V(G)$ with $|U| \geq k$ and $x \in V(G) - U$ are given. Let G' be obtained from G by adding a new vertex y adjacent to all vertices of U.

Fans and Fan Lemma

Let G be a graph, $U \subset V(G)$ and $x \in V(G) - U$. Then an x, U-fan of size k in G is a set of k paths from x to U such that any two of them share only x.

Theorem 4.15 (Fan Lemma): Let $n \geq k + 1$. An n-vertex graph G is k-connected if and only if for every choice $U \subset V(G)$ with $|U| \geq k$ and $x \in V(G) - U$, G has an x, U-fan of size k.

Proof. $(\Rightarrow)$ Suppose G is k-connected, and $U \subset V(G)$ with $|U| \geq k$ and $x \in V(G) - U$ are given. Let G' be obtained from G by adding a new vertex y adjacent to all vertices of U.

Since $|U| \geq k$, by Expansion Lemma, G' is also k-connected. So by Theorem 4.9 (or Theorem 4.11), G' has k int.-disjoint x, y-paths. When we remove y from each path, what remains is an x, U-fan of size k. This proves $(\Rightarrow)$.

$(\Leftarrow)$ We use contrapositive. Suppose G is not k-connected. Since $n \geq k+1$, there is an $S \subseteq V(G)$ with $|S| = k-1$ such that $G - S$ is disconnected. This means there is a partition $V(G) = S \cup A \cup B$ with $A \neq \emptyset$ and $B \neq \emptyset$ such that no edge connects A with B.

($\Leftarrow$) We use contrapositive. Suppose G is not k-connected. Since $n \geq k+1$, there is an $S \subseteq V(G)$ with $|S| = k-1$ such that $G - S$ is disconnected. This means there is a partition $V(G) = S \cup A \cup B$ with $A \neq \emptyset$ and $B \neq \emptyset$ such that no edge connects A with B.

Let $a \in A$ and $b \in B$. Take $U = S \cup B$ and $x = a$. Then each x, U-path in G contains a vertex of S. Since $|S| \leq k-1$, G has no x, U-fan of size k, as claimed.

Definitions

For a digraph $G = (V, E)$ and $v \in V$, let $E^+(v)$ denote the set of edges leaving v and $E^-(v)$ — the set of edges entering v.

A network $G = \{V, E, s, t, \mathbf{c} = \{c(e)\}_{e \in E}\}$ is a directed graph (V, E) with a source vertex s, a sink vertex t, and a set of non-negative capacities $\{c(e)\}_{e \in E}$ of edges.

A function $f : E \to \mathbf{R}$ is called a flow in G if for every vertex $v \in V - s - t$,

$$div_f(v) = \sum_{e \in E^+(v)} f(e) - \sum_{e \in E^-(v)} f(e) = 0. \tag{1}$$

Definitions

For a digraph $G = (V, E)$ and $v \in V$, let $E^+(v)$ denote the set of edges leaving v and $E^-(v)$ — the set of edges entering v.

A network $G = \{V, E, s, t, \mathbf{c} = \{c(e)\}_{e \in E}\}$ is a directed graph (V, E) with a source vertex s, a sink vertex t, and a set of non-negative capacities $\{c(e)\}_{e \in E}$ of edges.

A function $f : E \to \mathbf{R}$ is called a flow in G if for every vertex $v \in V - s - t$,

$$div_f(v) = \sum_{e \in E^+(v)} f(e) - \sum_{e \in E^-(v)} f(e) = 0. \tag{1}$$

If $0 \leq f(e) \leq \mathbf{c}(e)$ for every $e \in E$, then the flow is called feasible (for G).

Simple properties

Consider $\sum_{v \in V} div_f(v)$.
If f is a flow, then by (1) this sum equals $div_f(s) + div_f(t)$.

Simple properties

Consider $\sum_{v \in V} div_f(v)$.
If f is a flow, then by (1) this sum equals $div_f(s) + div_f(t)$.

On the other hand, every edge uv contributes $f(uv)$ to

$$div_f(u) = \sum_{e \in E^+(u)} f(e) - \sum_{e \in E^-(u)} f(e)$$

and $-f(uv)$ to

$$div_f(v) = \sum_{e \in E^+(v)} f(e) - \sum_{e \in E^-(v)} f(e).$$

Therefore, $div_f(s) + div_f(t) = 0$. $(*)$

Simple properties

Consider $\sum_{v \in V} div_f(v)$.
If f is a flow, then by (1) this sum equals $div_f(s) + div_f(t)$.

On the other hand, every edge uv contributes $f(uv)$ to

$$div_f(u) = \sum_{e \in E^+(u)} f(e) - \sum_{e \in E^-(u)} f(e)$$

and $-f(uv)$ to

$$div_f(v) = \sum_{e \in E^+(v)} f(e) - \sum_{e \in E^-(v)} f(e).$$

Therefore, $div_f(s) + div_f(t) = 0$. $(*)$

The value $M(f) = div_f(s) = -div_f(t)$ is called the value of f.
A flow with value zero is called circulation.

More definitions and a lemma

By definition, each flow f is a vector satisfying a system of linear equations and $M(f)$ is a linear function of this vector. So, for any flows f and g and any reals α and β,

$$M(\alpha f + \beta g) = \alpha M(f) + \beta M(g). \tag{2}$$

More definitions and a lemma

By definition, each flow f is a vector satisfying a system of linear equations and $M(f)$ is a linear function of this vector. So, for any flows f and g and any reals α and β,

$$M(\alpha f + \beta g) = \alpha M(f) + \beta M(g). \tag{2}$$

A flow f is positive if $f(e) \geq 0$ for every $e \in E$ and there exists $e_0 \in E$ such that $f(e_0) > 0$.
We say that a flow f is a flow along a (directed) cycle (or along a (directed) s,t-path or t,s-path) if f is non-zero only on the edges of this cycle (s,t-path or t,s-path) and $f(e) = f(e')$ for all e, e' in this cycle (s,t-path or t,s-path).

These flows are ”simplest possible”. By definition, a flow along a cycle is a circulation. It turns out that even most complicated flows are sums of these simple flows.

Lemma 4.15. Every positive circulation f in a network G can be represented as the sum of at most $|E(G)| - 1$ positive flows along cycles.

Lemma 4.15. Every positive circulation f in a network G can be represented as the sum of at most $|E(G)| - 1$ positive flows along cycles.

Proof. We use induction on the number m of the edges of G. The minimum possible number of edges is 2, and the only possible positive circulation with two edges is below.

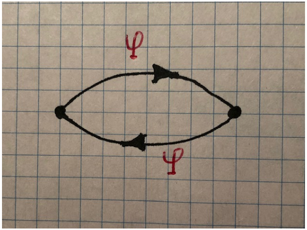


Figure: Circulation with 2 edges.

Suppose the lemma holds for all circulations in networks with fewer than m edges, and let f be a positive circulation in a network G with m edges.

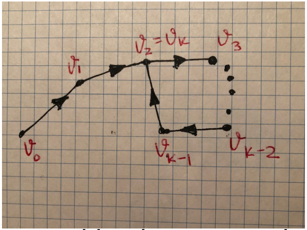
If $f(e_0) = 0$ for some $e_0 \in E$, then consider $G_0 = G - e_0$ and $f_0 = f\mid_{G_0}$. By the minimality of G, f_0 is the sum of at most $|E(G_0)| - 1 = m - 2$ positive flows along cycles. So, f is the sum of the same flows.

If $f(e_0) = 0$ for some $e_0 \in E$, then consider $G_0 = G - e_0$ and $f_0 = f\mid_{G_0}$. By the minimality of G, f_0 is the sum of at most $|E(G_0)| - 1 = m - 2$ positive flows along cycles. So, f is the sum of the same flows.

Thus we may assume that $f(e) > 0$ for every $e \in E$. Consider an arbitrary $e_1 = v_0 v_1 \in E$. Since f is a circulation, there is an edge $e_2 = v_1 v_2$ leaving v_1. Similarly, there exists an edge $e_3 = v_2 v_3$ leaving v_2, and so on.

If $f(e_0) = 0$ for some $e_0 \in E$, then consider $G_0 = G - e_0$ and $f_0 = f|_{G_0}$. By the minimality of G, f_0 is the sum of at most $|E(G_0)| - 1 = m - 2$ positive flows along cycles. So, f is the sum of the same flows.

Thus we may assume that $f(e) > 0$ for every $e \in E$. Consider an arbitrary $e_1 = v_0v_1 \in E$. Since f is a circulation, there is an edge $e_2 = v_1v_2$ leaving v_1. Similarly, there exists an edge $e_3 = v_2v_3$ leaving v_2, and so on.



Let k be the minimum positive integer such that $v_k \in \{v_0, \ldots, v_{k-1}\}$. For definiteness, let $v_k = v_s$. Then $C = v_s v_{s+1} \ldots v_{k-1} v_k$ is a cycle in G.

Let $\rho = \min\{f(e) \mid e \in E(C)\}$, and $\varphi(C, \rho)$ be the flow along C of size ρ.
Consider $f_1 = f - \varphi(C, \rho)$. If $f_1 \equiv 0$, then f is the sum of one flow along a cycle (namely, along C).

Let $\rho = \min\{f(e) \mid e \in E(C)\}$, and $\varphi(C, \rho)$ be the flow along C of size ρ.
Consider $f_1 = f - \varphi(C, \rho)$. If $f_1 \equiv 0$, then f is the sum of one flow along a cycle (namely, along C).

Otherwise, f_1 is a positive flow and there exists $e_1 \in E(C)$ with $f_1(e_1) = 0$.
Due to the minimality of G, the flow $f_1 |_{G-e_1}$ can be represented as the sum of at most $|E(G - e_1)| - 1 = |E(G)| - 2$ positive flows along cycles.

Let $\rho = \min\{f(e) \mid e \in E(C)\}$, and $\varphi(C, \rho)$ be the flow along C of size ρ.
Consider $f_1 = f - \varphi(C, \rho)$. If $f_1 \equiv 0$, then f is the sum of one flow along a cycle (namely, along C).

Otherwise, f_1 is a positive flow and there exists $e_1 \in E(C)$ with $f_1(e_1) = 0$.
Due to the minimality of G, the flow $f_1 |_{G-e_1}$ can be represented as the sum of at most $|E(G - e_1)| - 1 = |E(G)| - 2$ positive flows along cycles.

Adding $\varphi(C, \rho)$ to this sum, we find a representation for f, as claimed.
This proves the lemma.