Flows in networks

Lecture 25

Definitions

For a digraph $G = (V, E)$ and $v \in V$, let $E^+(v)$ denote the set of edges leaving v and $E^-(v)$ — the set of edges entering v.

A network $G = \{V, E, s, t, \mathbf{c} = \{c(e)\}_{e \in E}\}$ is a directed graph (V, E) with a source vertex s, a sink vertex t, and a set of non-negative capacities $\{c(e)\}_{e \in E}$ of edges.

A function $f : E \to \mathbf{R}$ is called a flow in G if for every vertex $v \in V - s - t$,

$$div_f(v) = \sum_{e \in E^+(v)} f(e) - \sum_{e \in E^-(v)} f(e) = 0. \qquad (1)$$

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If $0 \leq f(e) \leq \mathbf{c}(e)$ for every $e \in E$, then the flow is called feasible (for G).

Simple properties

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The value $M(f) = div_f(s) = -div_f(t)$ is called the value of f.
A flow with value zero is called circulation.

More definitions and a lemma

By definition, each flow f is a vector satisfying a system of linear equations and $M(f)$ is a linear function of this vector. So, for any flows f and g and any reals α and β,

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A flow f is positive if $f(e) \geq 0$ for every $e \in E$ and there exists $e_0 \in E$ such that $f(e_0) > 0$.
We say that a flow f is a flow along a (directed) cycle (or along a (directed) s, t-path or t, s-path) if f is non-zero only on the edges of this cycle (s, t-path or t, s-path) and $f(e) = f(e')$ for all e, e' in this cycle (s, t-path or t, s-path).

These flows are "simplest possible". By definition, a flow along a cycle is a circulation. It turns out that even most complicated flows are sums of these simple flows.

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Proof. We use induction on the number m of the edges of G. The minimum possible number of edges is 2, and the only possible positive circulation with two edges is below.

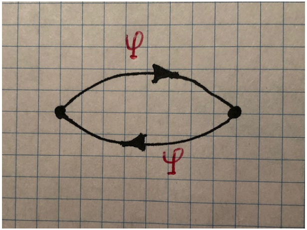

Figure: Circulation with 2 edges.

Suppose the lemma holds for all circulations in networks with fewer than m edges, and let f be a positive circulation in a network G with m edges.

If $f(e_0) = 0$ for some $e_0 \in E$, then consider $G_0 = G - e_0$ and $f_0 = f\,|_{G_0}$. By the minimality of G, f_0 is the sum of at most $|E(G_0)| - 1 = m - 2$ positive flows along cycles. So, f is the sum of the same flows.

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Thus we may assume that $f(e) > 0$ for every $e \in E$. Consider an arbitrary $e_1 = v_0 v_1 \in E$. Since f is a circulation, there is an edge $e_2 = v_1 v_2$ leaving v_1. Similarly, there exists an edge $e_3 = v_2 v_3$ leaving v_2, and so on.

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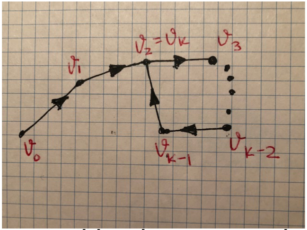

Let k be the minimum positive integer such that $v_k \in \{v_0, \ldots, v_{k-1}\}$. For definiteness, let $v_k = v_s$. Then $C = v_s v_{s+1} \ldots v_{k-1} v_k$ is a cycle in G.

Let $\rho = \min\{f(e) \mid e \in E(C)\}$, and $\varphi(C, \rho)$ be the flow along C of size ρ.

Consider $f_1 = f - \varphi(C, \rho)$. If $f_1 \equiv 0$, then f is the sum of one flow along a cycle (namely, along C).

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Otherwise, f_1 is a positive flow and there exists $e_1 \in E(C)$ with $f_1(e_1) = 0$.
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Adding $\varphi(C, \rho)$ to this sum, we find a representation for f, as claimed.
This proves the lemma.

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Then f_0 is a circulation in G_0. By Lemma 4.15, f_0 is the sum of at most $|E(G_0)| - 1 = |E(G)|$ positive flows along cycles, say along cycles $C_1, \ldots, C_m$ where $m \leq |E(G)|$. Delete edge e_0 from each C_i containing it. What remains in such a cycle is an s,t-path P_i. Thus f is the sum of the flows along paths P_is and along cycles C_js that do not contain e_0.

Case 2: $M(f) < 0$. Let G_1 be obtained from G by adding new edge $e_1 = st$ and let f_1 differ from f only in that $f_1(e_1) = -M(f)$. Then the argument is the same as in Case 1.

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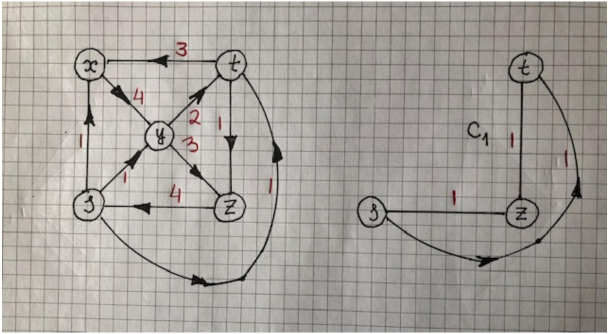
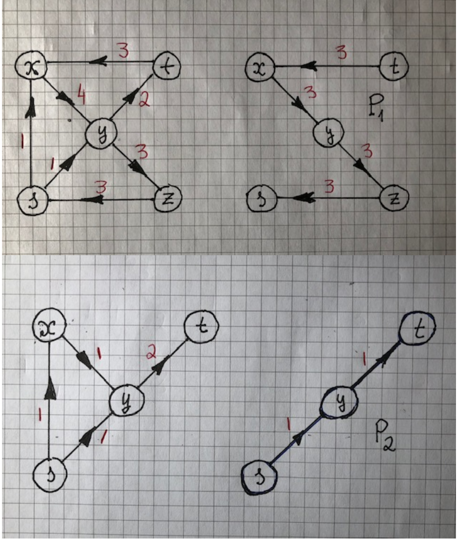
An example:





Hence $f = \phi(C_1, 1) + \phi(P_1, 3) + \phi(P_2, 1) + \phi(P_3, 1)$.