Connections, bipartite graphs

Lecture 3

Walks

A **walk** in a graph G is a list $v_0, e_1, v_1, e_2, v_2, \ldots, e_\ell, v_\ell$ of vertices v_i and edges e_i such that for each $1 \leq i \leq \ell$, the endpoints of e_i are v_{i-1} and v_i.

If the first vertex of a walk is u and the last vertex on the walk is v, we call this a u, v-walk. When G is a simple graph, we also may specify a walk by simply listing the vertices, since it is unambiguous which edge is traversed in each step.

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A u, v-**trail** is a u, v-walk with no repeated edges (but vertices may repeat). If $u \neq v$, a u, v-**path** is a u, v-walk with no repeated vertices.
(You should convince yourself that the subgraph definition of path matches up with the walk definition of a path).
If $u = v$, then we call a u, v-walk or trail **closed**. The **length** of a walk, trail or path is the number of edges traversed.

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Proof. We will proceed by induction on the length ℓ of the u, v-walk W.

If $\ell = 0$, then W has no edges, so $u = v$ and there are no repeated vertices or edges. Thus W is a u, v-path.

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Now assume $\ell > 0$ and all u, v-walks of length $\ell' < \ell$ contain a u, v-path. Let W be any u, v-walk of length ℓ. If W does not repeat any vertices, then W also cannot repeat any edges, so W is a path by itself. Thus, we may assume there is some vertex w that appears at least twice in our walk.

Then W has the form

$$(u = v_0, v_1, v_2, \dots, \mathbf{v_{k_1}} = \mathbf{w}, v_{k_1+1}, \dots, \mathbf{v_{k_2}} = \mathbf{w}, v_{k_2+1}, \dots, v_{\ell+1} = v)$$

for some choice of $k_1, k_2 \in \mathbb{N}$, $k_1 < k_2$.
(Note 1: $v = v_{\ell+1}$ because a walk of LENGTH ℓ contains $\ell + 1$ vertices.
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However, this implies that

$$W' = (u = v_0, v_1, v_2, \ldots, \mathbf{v_{k_1}} = \mathbf{v_{k_2}} = \mathbf{w}, v_{k_2+1}, \ldots, v_{\ell+1} = v)$$

is a subwalk of W of length $\ell' = \ell - (k_2 - k_1) < \ell$. So by our inductive hypothesis W' contains a u, v-path, and thus so does W. Thus, via induction, we have proved that every u, v-walk contains a u, v-path. □

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Lemma 1.2 (Lemma 1.2.15 in the book) : Every closed walk of odd length contains an odd cycle.

Proof. In Lecture 4.

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(2) If G is 2-regular, then each component of G is a cycle.

Proof. If G has a loop, this loop is already a cycle. Suppose, G is loopless. Consider a longest path, say $P = v_0, e_1, v_1, \ldots, e_k, v_k$ in G. Since $\delta(G) \geq 2$, there is an edge $e_0 \neq e_1$ incident to v_0. Let w be the other end of e_0.

By the maximality of P, $w \in \{v_1, \ldots, v_k\}$. If $w = v_i$, then we have cycle $v_0, e_1, v_1, \ldots, e_i, v_i, e_0, v_0$. This proves (1).

Let G be 2-regular and connected. Repeat the proof of (1): If G has a loop then since $\Delta(G) = 2$, G is a loop. So, suppose, G is loopless. Consider a longest path, say $P = v_0, e_1, v_1, \ldots, e_k, v_k$ in G. Since $\delta(G) = 2$, there is an edge $e_0 \neq e_1$ incident to v_0. Let w be the other end of e_0. Since $\Delta(G) = 2$, vertex w can be only v_k. □

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Construct sets $V_1, V_2, \ldots$ as follows.
Let $V_1 = \{v\}$. For $i = 1, 2, \ldots$, if $\bigcup_{j=1}^{i} V_j = V(G)$, then Stop, otherwise, let $V_{i+1} = N(V_i) - \bigcup_{j=1}^{i} V_j$.

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We claim that every V_i is independent.

The claim is trivial for $i = 1$. Suppose $i > 1$, $x, y \in V_i$ and $xy \in E(G)$. By the definition of V_i, G contains a path P_x from v to x and a path P_y from v to y, each with $i - 1$ edges. Then the walk obtained from P_x by adding edge xy and then the reverse of P_y is a closed walk if odd length. By Lemma 1.2, it contains and odd cycle, a contradiction. This proves the claim.

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Then the sets $S_1 = V_1 \cup V_3 \cup V_5 \cup \ldots$ and $S_2 = V_2 \cup V_4 \cup V_6 \cup \ldots$ are independent, and thus G is bipartite. □

Theorem 1.4 (Proposition 1.3.3) **Degree Sum Formula**: For every graph G,

$$\sum_{v \in V(G)} d(v) = 2|E(G)|.$$

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Theorem 1.4 (Proposition 1.3.3) **Degree Sum Formula**: For every graph G,

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Proof. Each edge has exactly two endpoints, and so contributes to the sum of degrees exactly twice. □