Plane graphs and planar graphs. Part 1

Lecture 30

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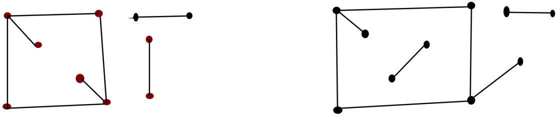
Two distinct plane graphs.



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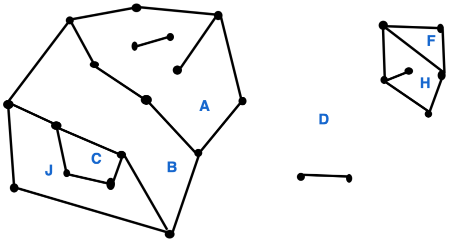

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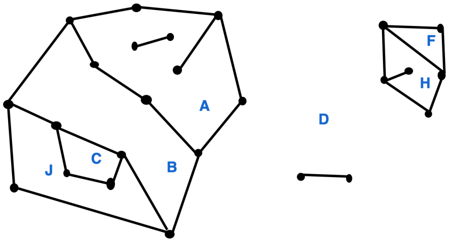

Definition of dual graphs: given in class (and book).

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Theorem 6.2 (Euler's Formula): For every connected plane graph (G, φ),

$$|V(G)| - |E(G)| + |F(G, \varphi)| = 2.$$

Proof of Euler's Formula

For given n, we use induction of $m = |E(G)|$.

Base of induction: $m = n - 1$. Let (G, φ) be a plane drawing of an n-vertex connected graph G with $n - 1$ edges. By the Characterization Theorem for trees, G is a tree.

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Since G has no cycles, (G, φ) has only one face. Hence,

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Induction step: Suppose the formula holds for all planar drawings of all n-vertex connected graphs with $m - 1$ edges. Let (G, φ) be a plane drawing of an n-vertex connected graph G with m edges.

Since $m \geq n$, G has a cycle, say C.
In drawing φ, C forms a closed simple polygonal curve. By Restricted Jordan Curve Theorem, C divides $\mathbf{R}^2$ into two components. Hence each face of (G, φ) is either outside of C or inside of C.

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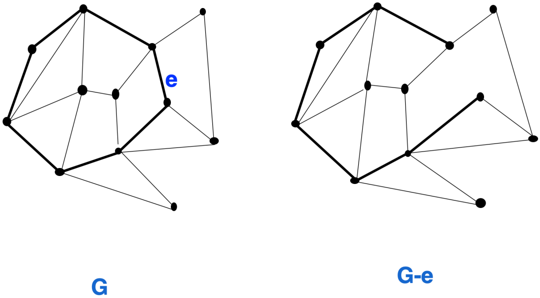
Let e be an edge of C. Let (G', φ') be obtained from (G, φ) by deleting e. Then the two faces of (G, φ) containing e on the boundary (one inside C and one outside of C) merge into one face of (G', φ').



Since e was not a cut edge, G' is connected. By the induction assumption,

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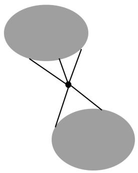
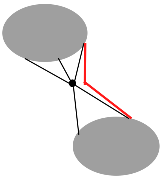
Corollary 6.4: Graphs K_5 and $K_{3,3}$ are not planar.

Proof of Corollary 6.3

It is enough to prove the corollary for connected planar simple graphs. For the main part, even 2-connected.





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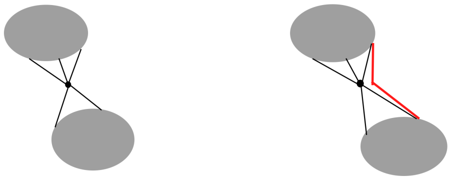

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Let φ be a planar drawing of G. By Euler's Formula and Claim 1,

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Again, let φ be a planar drawing of our triangle-free G. By Euler's Formula and Claim 2,

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Proof of Corollary 6.4. $|V(K_5)| = 5$ and $|E(K_5)| = 10 = 3|V(K_5)| - 5$. So, by Cor. 6.3, K_5 is not planar.

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Similarly, $K_{3,3}$ is triangle-free, $|V(K_{3,3})| = 6$ and $|E(K_{3,3})| = 9 = 2|V(K_{3,3})| - 3$.
Again, by Cor. 6.3, $K_{3,3}$ is not planar.

Subdivisions

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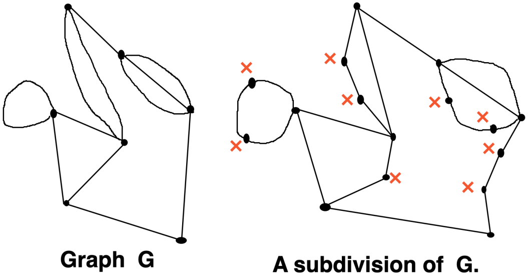

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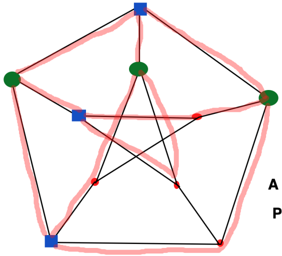

A subdivision of K_{3,3} in Petersen graph.

Note that even after deleting two edges from the Petersen graph, the remaining graph is not planar.