Plane graphs and planar graphs. Part 2

Lecture 31

By $F(G, \varphi)$ we denote the set of faces of the plane graph (G, φ).

Proposition 6.1: For each plane graph (G, φ),

$$\sum_{F_i \in F(G,\varphi)} \ell(F_i) = 2|E(G)|. \qquad (1)$$

Theorem 6.2 (Euler's Formula): For every connected plane graph (G, φ),

$$|V(G)| - |E(G)| + |F(G, \varphi)| = 2.$$

Observation: A graph is planar if and only if it can be drawn on the sphere.

By $F(G, \varphi)$ we denote the set of faces of the plane graph (G, φ).

Proposition 6.1: For each plane graph (G, φ),

$$\sum_{F_i \in F(G,\varphi)} \ell(F_i) = 2|E(G)|. \tag{2}$$

Corollary 6.3: For $n \geq 3$, every simple planar n-vertex graph G has at most $3n - 6$ edges. Moreover, if G is triangle-free, then G has at most $2n - 4$ edges.

Corollary 6.4: Graphs K_5 and $K_{3,3}$ are not planar.

Proof of Corollary 6.3

It is enough to prove the corollary for connected planar simple graphs.

Claim 1: For each planar drawing φ of G,

$$3|F(G,\varphi)| \leq 2|E(G)|. \tag{3}$$

Proof: Since $\ell(F_i) \geq 3$ for each F_i, by Proposition 6.1,

$$2|E(G)| = \sum_{F_i \in F(G,\varphi)} \ell(F_i) \geq 3|F(G,\varphi)|.$$

Let φ be a planar drawing of G. By Euler's Formula and Claim 1,

$$2 = |V(G) - |E(G)| + |F(G,\varphi)| \leq |V(G)| - |E(G)| + \frac{2}{3}|E(G)|.$$

Hence

$$\frac{1}{3}|E(G)| \leq |V(G)| - 2, \qquad \text{as claimed.}$$

Hence

$$\frac{1}{3}|E(G)| \leq |V(G)| - 2, \qquad \text{as claimed.}$$

For the "Moreover" part, we modify the claim:
Claim 2: For each planar drawing φ of triangle-free G,

$$|F(G, \varphi)| \leq \frac{|E(G)|}{2}. \tag{4}$$

Hence

$$\frac{1}{3}|E(G)| \le |V(G)| - 2, \qquad \text{as claimed.}$$

For the "Moreover" part, we modify the claim:

Claim 2: For each planar drawing φ of triangle-free G,

$$|F(G,\varphi)| \le \frac{|E(G)|}{2}. \tag{4}$$

Proof: Since $\ell(F_i) \ge 4$ for each F_i, by Proposition 6.1,

$$2|E(G)| = \sum_{F_i \in F(G,\varphi)} \ell(F_i) \ge 4|F(G,\varphi)|.$$

Again, let φ be a planar drawing of our triangle-free G. By Euler's Formula and Claim 2,

$$2 = |V(G) - |E(G)| + |F(G, \varphi)| \leq |V(G)| - |E(G)| + \frac{1}{2}|E(G)|.$$

Again, let φ be a planar drawing of our triangle-free G. By Euler's Formula and Claim 2,

$$2 = |V(G) - |E(G)| + |F(G,\varphi)| \leq |V(G)| - |E(G)| + \frac{1}{2}|E(G)|.$$

Hence

$$\frac{1}{2}|E(G)| \leq |V(G)| - 2, \qquad \text{as claimed.}$$

Again, let φ be a planar drawing of our triangle-free G. By Euler's Formula and Claim 2,

$$2 = |V(G) - |E(G)| + |F(G, \varphi)| \leq |V(G)| - |E(G)| + \frac{1}{2}|E(G)|.$$

Hence

$$\frac{1}{2}|E(G)| \leq |V(G)| - 2, \qquad \text{as claimed.}$$

Questions: 1. What happens for $|V(G)| \leq 2$?
2. In which part(s) of the proof have I used $|V(G)| \geq 3$?

Again, let φ be a planar drawing of our triangle-free G. By Euler's Formula and Claim 2,

$$2 = |V(G) - |E(G)| + |F(G,\varphi)| \leq |V(G)| - |E(G)| + \frac{1}{2}|E(G)|.$$

Hence

$$\frac{1}{2}|E(G)| \leq |V(G)| - 2, \qquad \text{as claimed.}$$

Questions: 1. What happens for $|V(G)| \leq 2$?
2. In which part(s) of the proof have I used $|V(G)| \geq 3$?

Proof of Corollary 6.4. $|V(K_5)| = 5$ and $|E(K_5)| = 10 = 3|V(K_5)| - 5$. So, by Cor. 6.3, K_5 is not planar.

Similarly, $K_{3,3}$ is triangle-free, $|V(K_{3,3})| = 6$ and $|E(K_{3,3})| = 9 = 2|V(K_{3,3})| - 3$.
Again, by Cor. 6.3, $K_{3,3}$ is not planar.

Subdivisions

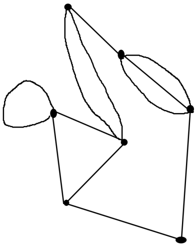
A subdivision of an edge e connecting vertices x and y is a replacement of e with the path x, z, y, where z is a new vertex.

A graph H is a subdivision of a graph G if H can be obtained from G by series of subdivisions of edges.

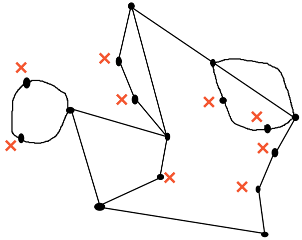
Subdivisions

A subdivision of an edge e connecting vertices x and y is a replacement of e with the path x, z, y, where z is a new vertex.

A graph H is a subdivision of a graph G if H can be obtained from G by series of subdivisions of edges.



Graph G



A subdivision of G.

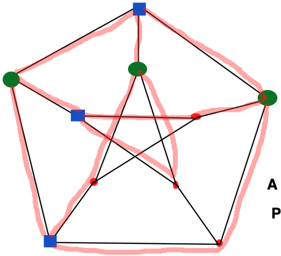
Observation: Let H be a subdivision of a graph G. Then H is planar if and only if G is planar.

Observation: Let H be a subdivision of a graph G. Then H is planar if and only if G is planar.

Theorem 6.5 (Kuratowski's Theorem) A graph G is planar if and only if G does not contain subdivisions of K_5 and $K_{3,3}$.

Observation: Let H be a subdivision of a graph G. Then H is planar if and only if G is planar.

Theorem 6.5 (Kuratowski's Theorem) A graph G is planar if and only if G does not contain subdivisions of K_5 and $K_{3,3}$.



A subdivision of K_{3,3} in Petersen graph.

Note that even after deleting two edges from the Petersen graph, the remaining graph is not planar.

Contractions

A **contraction** of an edge xy in a simple graph G is a replacement of the vertices x and y with a new vertex $x * y$ whose neighborhood is $N(x) \cup N(y) - x - y$.

Contractions

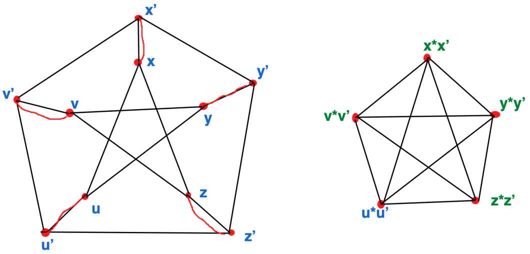
A **contraction** of an edge xy in a simple graph G is a replacement of the vertices x and y with a new vertex $x * y$ whose neighborhood is $N(x) \cup N(y) - x - y$.

A graph H is a minor of graph G, if H can be obtained from G by a sequence of the following operations:
(a) deleting a vertex, (b) deleting an edge,
(c) contracting an edge.

Contractions

A **contraction** of an edge xy in a simple graph G is a replacement of the vertices x and y with a new vertex $x * y$ whose neighborhood is $N(x) \cup N(y) - x - y$.

A graph H is a minor of graph G, if H can be obtained from G by a sequence of the following operations:
(a) deleting a vertex, (b) deleting an edge,
(c) contracting an edge.



Contracting Petersen graph to K_5.

Observation: Let H be a planar graph. Then after contracting any edge the resulting graph is planar.

Observation: Let H be a planar graph. Then after contracting any edge the resulting graph is planar.

Theorem 6.6 (Wagner): A graph G is planar if and only if G does not contain K_5 and $K_{3,3}$ as minors.

Observation: Let H be a planar graph. Then after contracting any edge the resulting graph is planar.

Theorem 6.6 (Wagner): A graph G is planar if and only if G does not contain K_5 and $K_{3,3}$ as minors.

Proof. We use contrapositive in both directions. Assume first that G contains a minor H of K_5 or $K_{3,3}$. Since H is not planar, by the observation above, G is also not planar.

Observation: Let H be a planar graph. Then after contracting any edge the resulting graph is planar.

Theorem 6.6 (Wagner): A graph G is planar if and only if G does not contain K_5 and $K_{3,3}$ as minors.

Proof. We use contrapositive in both directions. Assume first that G contains a minor H of K_5 or $K_{3,3}$. Since H is not planar, by the observation above, G is also not planar.

Suppose now that G is not planar. Then by **Theorem 6.5**, G contains a subdivision G' of $H \in \{K_5, K_{3,3}\}$. But the fact that G' is a subdivision of H implies that H is a minor of G'. Since each subgraph of G is a minor of G, our graph $H \in \{K_5, K_{3,3}\}$ is a minor of G.

Outerplanar and outerplane graphs

A pair (G, φ) forms an outerplane graph if all the vertices of G are on the boundary of the unbounded face.

Outerplanar and outerplane graphs

A pair (G, φ) forms an outerplane graph if all the vertices of G are on the boundary of the unbounded face.

A graph G is outerplanar if it has a drawing φ such that the pair (G, φ) forms an outerplane graph.

Outerplanar and outerplane graphs

A pair (G, φ) forms an outerplane graph if all the vertices of G are on the boundary of the unbounded face.

A graph G is outerplanar if it has a drawing φ such that the pair (G, φ) forms an outerplane graph.

Theorem 6.7: A graph G is outerplanar if and only if G does not contain subdivisions of K_4 and $K_{2,3}$.

Outerplanar and outerplane graphs

A pair (G, φ) forms an outerplane graph if all the vertices of G are on the boundary of the unbounded face.

A graph G is outerplanar if it has a drawing φ such that the pair (G, φ) forms an outerplane graph.

Theorem 6.7: A graph G is outerplanar if and only if G does not contain subdivisions of K_4 and $K_{2,3}$.

Proof. Let G^* be obtained from G by adding a new vertex y adjacent to each vertex of G.

Outerplanar and outerplane graphs

A pair (G, φ) forms an outerplane graph if all the vertices of G are on the boundary of the unbounded face.

A graph G is outerplanar if it has a drawing φ such that the pair (G, φ) forms an outerplane graph.

Theorem 6.7: A graph G is outerplanar if and only if G does not contain subdivisions of K_4 and $K_{2,3}$.

Proof. Let G^* be obtained from G by adding a new vertex y adjacent to each vertex of G.

Claim 1: G^* is planar if and only if G is outerplanar.

Outerplanar and outerplane graphs

A pair (G, φ) forms an outerplane graph if all the vertices of G are on the boundary of the unbounded face.

A graph G is outerplanar if it has a drawing φ such that the pair (G, φ) forms an outerplane graph.

Theorem 6.7: A graph G is outerplanar if and only if G does not contain subdivisions of K_4 and $K_{2,3}$.

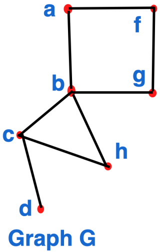
Proof. Let G^* be obtained from G by adding a new vertex y adjacent to each vertex of G.

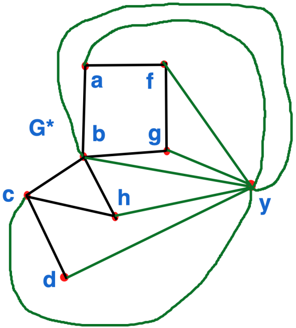
Claim 1: G^* is planar if and only if G is outerplanar.

Proof of Claim 1. If G^* is planar, draw it so that y is on the outer face. Delete y. In the obtained drawing of G, all vertices are on the outer face.

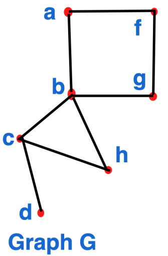
If G is outerplanar, let φ be a drawing of G such that all vertices are on the boundary of the outer face. Then we can draw y in the outer face and connect it to all vertices of G so that we get a planar drawing of G^*. This proves the claim.

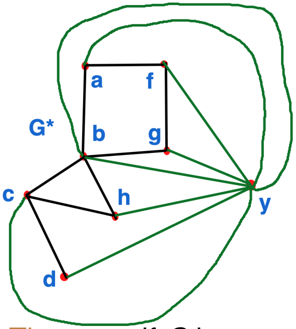
If G is outerplanar, let φ be a drawing of G such that all vertices are on the boundary of the outer face. Then we can draw y in the outer face and connect it to all vertices of G so that we get a planar drawing of G^*. This proves the claim.





If G is outerplanar, let φ be a drawing of G such that all vertices are on the boundary of the outer face. Then we can draw y in the outer face and connect it to all vertices of G so that we get a planar drawing of G^*. This proves the claim.





Thus by Claim 1 and Kuratowski's Theorem, if G is outerplanar, then G^* does not contain subdivisions of K_5 and $K_{3,3}$. But then $G = G^* - y$ cannot contain a subdivision of K_4 or $K_{2,3}$.

On the other hand, by the same claim and the same theorem, if G is not outerplanar, then G^* contains a subdivision of K_5 or of $K_{3,3}$. In the first case, $G = G^* - y$ contains a subdivision of K_4. In the second case, it contains a subdivision of $K_{2,3}$.
This proves Theorem 6.7.

On the other hand, by the same claim and the same theorem, if G is not outerplanar, then G^* contains a subdivision of K_5 or of $K_{3,3}$. In the first case, $G = G^* - y$ contains a subdivision of K_4. In the second case, it contains a subdivision of $K_{2,3}$.
This proves Theorem 6.7.

A simple graph is maximal planar if it is planar but adding any non-loop edge not parallel to any edge of G results in a nonplanar graph.

On the other hand, by the same claim and the same theorem, if G is not outerplanar, then G^* contains a subdivision of K_5 or of $K_{3,3}$. In the first case, $G = G^* - y$ contains a subdivision of K_4. In the second case, it contains a subdivision of $K_{2,3}$.
This proves Theorem 6.7.

A simple graph is maximal planar if it is planar but adding any non-loop edge not parallel to any edge of G results in a nonplanar graph.

A triangulation is a plane graph (G, φ) such that every face of it is a 3-cycle.

On the other hand, by the same claim and the same theorem, if G is not outerplanar, then G^* contains a subdivision of K_5 or of $K_{3,3}$. In the first case, $G = G^* - y$ contains a subdivision of K_4. In the second case, it contains a subdivision of $K_{2,3}$.
This proves Theorem 6.7.

A simple graph is maximal planar if it is planar but adding any non-loop edge not parallel to any edge of G results in a nonplanar graph.

A triangulation is a plane graph (G, φ) such that every face of it is a 3-cycle.

Theorem 6.8: Let $n \geq 3$. For a simple n-vertex plane graph (G, φ), TFAE:
(A) G has $3n - 6$ edges;
(B) (G, φ) is a triangulation;
(C) G is maximal planar.

Proof of Theorem 6.8:

(A) $\Rightarrow$ (C). By Cor. 6.3, a simple n-vertex planar graph cannot have $3n - 5$ edges. Thus (A) implies (C).

Proof of Theorem 6.8:

(A) $\Rightarrow$ (C). By Cor. 6.3, a simple n-vertex planar graph cannot have $3n - 5$ edges. Thus (A) implies (C).

(B) $\Rightarrow$ (A). If (G, φ) is a triangulation, then $3|F(G, \varphi)| = 2|E(G)|$. Plugging this into Euler's Formula, we get

$$2 = n - |E(G)| + \frac{2}{3}|E(G)| = n - \frac{1}{3}|E(G)|,$$

which is equivalent to $|E(G)| = 3(n - 2) = 3n - 6$.

Proof of Theorem 6.8:

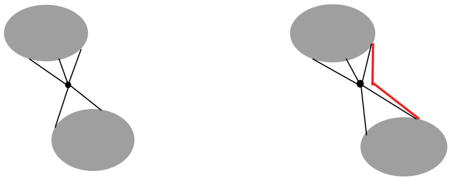
(A) $\Rightarrow$ (C). By Cor. 6.3, a simple n-vertex planar graph cannot have $3n - 5$ edges. Thus (A) implies (C).

(B) $\Rightarrow$ (A). If (G, φ) is a triangulation, then $3|F(G, \varphi)| = 2|E(G)|$. Plugging this into Euler's Formula, we get

$$2 = n - |E(G)| + \frac{2}{3}|E(G)| = n - \frac{1}{3}|E(G)|,$$

which is equivalent to $|E(G)| = 3(n-2) = 3n - 6$.

(C) $\Rightarrow$ (B). Let G be maximal planar and φ be a drawing of G. As in the proof of Corollary 6.3, G is 2-connected.

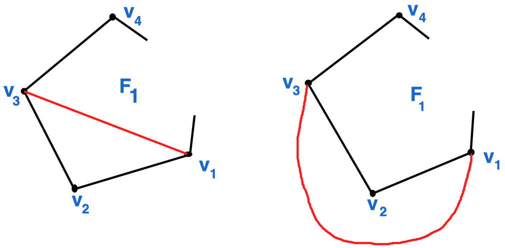


Finishing proof of Theorem 6.8

Since G is 2-connected, the boundary of each face is a cycle. Suppose the boundary of some face F_1 of (G, φ) is a cycle $v_1, v_2, \ldots, v_k$ for some $k \geq 4$.

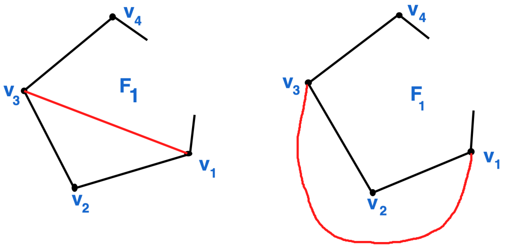
Finishing proof of Theorem 6.8

Since G is 2-connected, the boundary of each face is a cycle.
Suppose the boundary of some face F_1 of (G, φ) is a cycle $v_1, v_2, \ldots, v_k$ for some $k \geq 4$.
Then we try to add the edge $v_1 v_3$ inside F_1.



Finishing proof of Theorem 6.8

Since G is 2-connected, the boundary of each face is a cycle.
Suppose the boundary of some face F_1 of (G, φ) is a cycle $v_1, v_2, \ldots, v_k$ for some $k \geq 4$.
Then we try to add the edge $v_1 v_3$ inside F_1.



The only possibility that we fail is that G already has edge $v_1 v_3$.
In this case, G has no edge $v_2 v_4$, and we can add this edge inside F_1, a contradiction to the maximality of G.

Main results in Chapter 6:

1. Euler's Formula.

2. Kuratowski's Theorem.