Graph coloring. Part 1

Lecture 32

A simple graph is maximal planar if it is planar but adding any non-loop edge not parallel to any edge of G results in a nonplanar graph.

A simple graph is maximal planar if it is planar but adding any non-loop edge not parallel to any edge of G results in a nonplanar graph.

A triangulation is a plane graph (G, φ) such that every face of it is a 3-cycle.

A simple graph is maximal planar if it is planar but adding any non-loop edge not parallel to any edge of G results in a nonplanar graph.

A triangulation is a plane graph (G, φ) such that every face of it is a 3-cycle.

Theorem 6.8: Let $n \geq 3$. For a simple n-vertex plane graph (G, φ), TFAE:
(A) G has $3n - 6$ edges;
(B) (G, φ) is a triangulation;
(C) G is maximal planar.

A simple graph is maximal planar if it is planar but adding any non-loop edge not parallel to any edge of G results in a nonplanar graph.

A triangulation is a plane graph (G, φ) such that every face of it is a 3-cycle.

Theorem 6.8: Let $n \geq 3$. For a simple n-vertex plane graph (G, φ), TFAE:
(A) G has $3n - 6$ edges;
(B) (G, φ) is a triangulation;
(C) G is maximal planar.

Proof of Theorem 6.8:
(A) $\Rightarrow$ (C). By Cor. 6.3, a simple n-vertex planar graph cannot have $3n - 5$ edges. Thus (A) implies (C).

(B) $\Rightarrow$ (A). If (G, φ) is a triangulation, then $3|F(G, \varphi)| = 2|E(G)|$. Plugging this into Euler's Formula, we get

$$2 = n - |E(G)| + \frac{2}{3}|E(G)| = n - \frac{1}{3}|E(G)|,$$

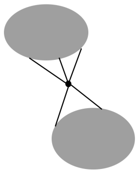
which is equivalent to $|E(G)| = 3(n-2) = 3n - 6$.

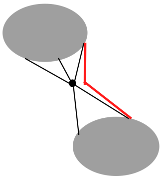
(B) $\Rightarrow$ (A). If (G, φ) is a triangulation, then $3|F(G, \varphi)| = 2|E(G)|$. Plugging this into Euler's Formula, we get

$$2 = n - |E(G)| + \frac{2}{3}|E(G)| = n - \frac{1}{3}|E(G)|,$$

which is equivalent to $|E(G)| = 3(n-2) = 3n-6$.

(C) $\Rightarrow$ (B). Let G be maximal planar and φ be a drawing of G. As in the proof of Corollary 6.3, G is 2-connected.



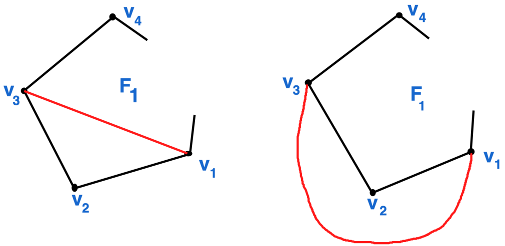


Finishing proof of Theorem 6.8

Since G is 2-connected, the boundary of each face is a cycle. Suppose the boundary of some face F_1 of (G, φ) is a cycle $v_1, v_2, \ldots, v_k$ for some $k \geq 4$.

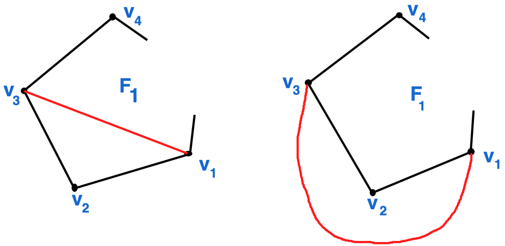
Finishing proof of Theorem 6.8

Since G is 2-connected, the boundary of each face is a cycle.
Suppose the boundary of some face F_1 of (G, φ) is a cycle $v_1, v_2, \ldots, v_k$ for some $k \geq 4$.
Then we try to add the edge $v_1 v_3$ inside F_1.



Finishing proof of Theorem 6.8

Since G is 2-connected, the boundary of each face is a cycle.
Suppose the boundary of some face F_1 of (G, φ) is a cycle $v_1, v_2, \ldots, v_k$ for some $k \geq 4$.
Then we try to add the edge $v_1 v_3$ inside F_1.



The only possibility that we fail is that G already has edge $v_1 v_3$.
In this case, G has no edge $v_2 v_4$, and we can add this edge inside F_1, a contradiction to the maximality of G.

Main results in Chapter 6:

1. Euler's Formula.

2. Kuratowski's Theorem.

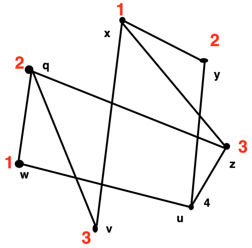
Definitions

A (proper) k-coloring of the vertices of a graph G is a mapping $f : V(G) \to \{1, \dots, k\}$ such that

$$f(x) \neq f(y) \qquad \forall\, e \in E(G) \text{ with ends } x \text{ and } y. \qquad (1)$$

Observations: 1. If G has a loop, then it has no k-coloring for any k.

2. Multiple edges do not affect coloring. So below we consider colorings only simple graphs.



Observation: Given a k-coloring f of the vertices of a graph G, for each $i \in \{1, \dots, k\}$, $f^{-1}(i)$ is an independent set. We call $f^{-1}(i)$ a color class of f.

So, a k-coloring of the vertices of a graph G is a partition of $V(G)$ into k independent sets.

Observation: Given a k-coloring f of the vertices of a graph G, for each $i \in \{1, \dots, k\}$, $f^{-1}(i)$ is an independent set. We call $f^{-1}(i)$ a color class of f.
So, a k-coloring of the vertices of a graph G is a partition of $V(G)$ into k independent sets.

The larger is k, the more freedom we have. We always can color the vertices of an n-vertex graphs with n colors.
The chromatic number, $\chi(G)$, of a graph G is the minimum positive integer k s.t. G has a k-coloring.

G is k-colorable if $\chi(G) \leq k$.

Question: Which graphs are 2-colorable?

Observation: Given a k-coloring f of the vertices of a graph G, for each $i \in \{1, \ldots, k\}$, $f^{-1}(i)$ is an independent set. We call $f^{-1}(i)$ a color class of f.
So, a k-coloring of the vertices of a graph G is a partition of $V(G)$ into k independent sets.

The larger is k, the more freedom we have. We always can color the vertices of an n-vertex graphs with n colors.
The chromatic number, $\chi(G)$, of a graph G is the minimum positive integer k s.t. G has a k-coloring.

G is k-colorable if $\chi(G) \leq k$.

Question: Which graphs are 2-colorable?

Fact: For each $k \geq 3$, the problem to check whether a graph G is k-colorable is NP-complete.

Examples and a simple fact

1. Complete graphs. 2. Cycles. 3. Petersen graph. 4. Wheels.

Examples and a simple fact

1. Complete graphs. 2. Cycles. 3. Petersen graph. 4. Wheels.

5. Consider the following 5 jobs: (a) Do an HW in Math 412, (b) Workout on a treadmill, (c) Eat a lunch, (d) Watch an episode of a TV series, (e) Clean the room.

Examples and a simple fact

1. Complete graphs. 2. Cycles. 3. Petersen graph. 4. Wheels.

5. Consider the following 5 jobs: (a) Do an HW in Math 412, (b) Workout on a treadmill, (c) Eat a lunch, (d) Watch an episode of a TV series, (e) Clean the room.

The clique number, $\omega(G)$, is the size of a largest clique (complete subgraph) in G.

Examples and a simple fact

1. Complete graphs. 2. Cycles. 3. Petersen graph. 4. Wheels.

5. Consider the following 5 jobs: (a) Do an HW in Math 412, (b) Workout on a treadmill, (c) Eat a lunch, (d) Watch an episode of a TV series, (e) Clean the room.

The clique number, $\omega(G)$, is the size of a largest clique (complete subgraph) in G.

Proposition 5.1. For every graph G,

$$\chi(G) \geq \omega(G) \quad \text{and} \quad \chi(G) \geq \frac{|V(G)|}{\alpha(G)}. \tag{2}$$

Examples and a simple fact

1. Complete graphs. 2. Cycles. 3. Petersen graph. 4. Wheels.

5. Consider the following 5 jobs: (a) Do an HW in Math 412, (b) Workout on a treadmill, (c) Eat a lunch, (d) Watch an episode of a TV series, (e) Clean the room.

The clique number, $\omega(G)$, is the size of a largest clique (complete subgraph) in G.

Proposition 5.1. For every graph G,

$$\chi(G) \geq \omega(G) \quad \text{and} \quad \chi(G) \geq \frac{|V(G)|}{\alpha(G)}. \tag{2}$$

Proof. All vertices in a clique of size $\omega(G)$ must have different colors. This proves $\chi(G) \geq \omega(G)$.

Examples and a simple fact

1. Complete graphs. 2. Cycles. 3. Petersen graph. 4. Wheels.

5. Consider the following 5 jobs: (a) Do an HW in Math 412, (b) Workout on a treadmill, (c) Eat a lunch, (d) Watch an episode of a TV series, (e) Clean the room.

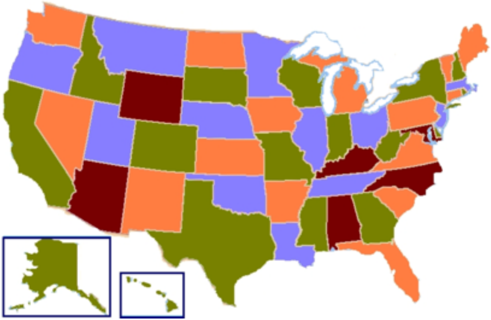
The clique number, $\omega(G)$, is the size of a largest clique (complete subgraph) in G.

Proposition 5.1. For every graph G,

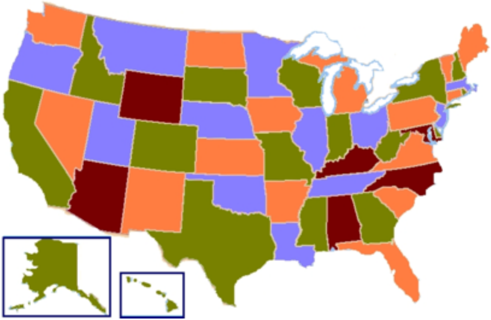
$$\chi(G) \geq \omega(G) \quad \text{and} \quad \chi(G) \geq \frac{|V(G)|}{\alpha(G)}. \tag{2}$$

Proof. All vertices in a clique of size $\omega(G)$ must have different colors. This proves $\chi(G) \geq \omega(G)$.
With any color, we can color at most $\alpha(G)$ vertices. This proves $\chi(G) \geq |V(G)|/\alpha(G)$.

Four Color Theorem



Four Color Theorem



The Four Color Theorem was proved (using computer verification) at the University of Illinois, in Altgeld Hall by K. Appel and W. Haken in 1976.

Greedy coloring

A greedy coloring of a graph G is defined as follows.

1. Order the vertices of G as, say $v_1, \ldots, v_n$.

Greedy coloring

A greedy coloring of a graph G is defined as follows.
1. Order the vertices of G as, say $v_1, \ldots, v_n$.

2. For $i = 1, 2, \ldots, n$ color v_i with the smallest positive integer distinct from the colors of the neighbors v_j of v_i with $j < i$.

Greedy coloring

A greedy coloring of a graph G is defined as follows.
1. Order the vertices of G as, say $v_1, \ldots, v_n$.

2. For $i = 1, 2, \ldots, n$ color v_i with the smallest positive integer distinct from the colors of the neighbors v_j of v_i with $j < i$.

Proposition 5.2. For every graph G, $\chi(G) \leq 1 + \Delta(G)$.

Greedy coloring

A greedy coloring of a graph G is defined as follows.
1. Order the vertices of G as, say $v_1, \ldots, v_n$.

2. For $i = 1, 2, \ldots, n$ color v_i with the smallest positive integer distinct from the colors of the neighbors v_j of v_i with $j < i$.

Proposition 5.2. For every graph G, $\chi(G) \leq 1 + \Delta(G)$.

Proof. Apply greedy coloring to G. At every Step i, at most $\Delta(G)$ colors are forbidden for v_i. So, there always is a color in $\{1, \ldots, 1 + \Delta(G)\}$ available to color v_i.

Greedy coloring

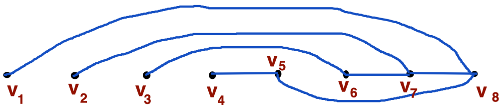
A greedy coloring of a graph G is defined as follows.
1. Order the vertices of G as, say $v_1, \ldots, v_n$.

2. For $i = 1, 2, \ldots, n$ color v_i with the smallest positive integer distinct from the colors of the neighbors v_j of v_i with $j < i$.

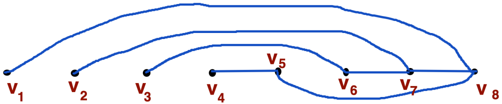
Proposition 5.2. For every graph G, $\chi(G) \leq 1 + \Delta(G)$.

Proof. Apply greedy coloring to G. At every Step i, at most $\Delta(G)$ colors are forbidden for v_i. So, there always is a color in $\{1, \ldots, 1 + \Delta(G)\}$ available to color v_i.

On the other hand, on the next slide we will see an example of a tree T_4 with an ordering of its vertices s.t. the greedy coloring of T_4 w.r.t. this ordering needs 4 colors. It is clear how to generalize this to a tree that will need a 1000 colors for its greedy coloring.

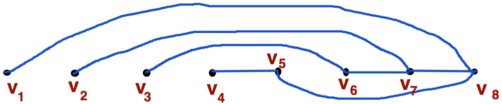


v_1
v_2
v_3
v_4
v_5
v_6
v_7
v_8



Definition A. A graph G is d-degenerate if for every subgraph H of G, $\delta(H) \leq d$.

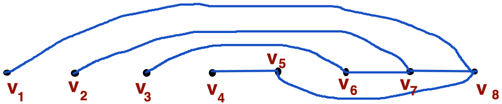
Example 1: A graph G is 1-degenerate iff G is a forest.



Definition A. A graph G is d-degenerate if for every subgraph H of G, $\delta(H) \leq d$.

Example 1: A graph G is 1-degenerate iff G is a forest.

Example 2: Every planar simple graph is 5-degenerate.

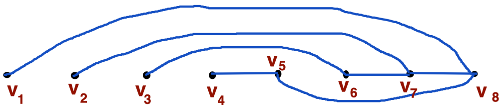


Definition A. A graph G is d-degenerate if for every subgraph H of G, $\delta(H) \leq d$.

Example 1: A graph G is 1-degenerate iff G is a forest.

Example 2: Every planar simple graph is 5-degenerate.

Definition B. A graph G is d-degenerate if its vertices can be ordered $v_1, \ldots, v_n$ so that for each $1 < i \leq n$, vertex v_i has at most d neighbors in $\{v_1, \ldots, v_{i-1}\}$.



Definition A. A graph G is d-degenerate if for every subgraph H of G, $\delta(H) \leq d$.

Example 1: A graph G is 1-degenerate iff G is a forest.

Example 2: Every planar simple graph is 5-degenerate.

Definition B. A graph G is d-degenerate if its vertices can be ordered $v_1, \ldots, v_n$ so that for each $1 < i \leq n$, vertex v_i has at most d neighbors in $\{v_1, \ldots, v_{i-1}\}$.

Proposition 5.3. Definitions (A) and (B) are equivalent.

Proof: In the lecture.

Proposition 5.4. Every d-degenerate graph is $(d+1)$-colorable.

Proof: Use the ordering of vertices provided by Definition B, and apply to it the greedy coloring.

Proposition 5.4. Every d-degenerate graph is $(d+1)$-colorable.

Proof: Use the ordering of vertices provided by Definition B, and apply to it the greedy coloring.

Theorem 5.5. Every planar graph is 6-colorable.

Proof: Let G be a planar graph. By Example 2, G is 5-degenerate. So by Proposition 5.4, G is 6-colorable.

Proposition 5.4. Every d-degenerate graph is $(d+1)$-colorable.

Proof: Use the ordering of vertices provided by Definition B, and apply to it the greedy coloring.

Theorem 5.5. Every planar graph is 6-colorable.

Proof: Let G be a planar graph. By Example 2, G is 5-degenerate. So by Proposition 5.4, G is 6-colorable.

For $k \geq 1$, a graph G is k-critical, if $\chi(G) = k$, but for each proper subgraph G' of G,

$$\chi(G') \leq k - 1.$$

Proposition 5.4. Every d-degenerate graph is $(d+1)$-colorable.

Proof: Use the ordering of vertices provided by Definition B, and apply to it the greedy coloring.

Theorem 5.5. Every planar graph is 6-colorable.

Proof: Let G be a planar graph. By Example 2, G is 5-degenerate. So by Proposition 5.4, G is 6-colorable.

For $k \geq 1$, a graph G is k-critical, if $\chi(G) = k$, but for each proper subgraph G' of G,

$$\chi(G') \leq k - 1.$$

Examples: (a) Complete graphs, (b) Odd cycles, (c) Odd wheels, (d) Moser spindle.

Theorem 5.6. Let $k \geq 3$ and G be a k-critical graph. Then
(a) $\kappa(G) \geq 2$;
(b) $\kappa'(G) \geq k - 1$.
Moreover, for each $k \geq 3$ there are infinitely many k-critical graphs with connectivity exactly 2.

Theorem 5.6. Let $k \geq 3$ and G be a k-critical graph. Then
(a) $\kappa(G) \geq 2$;
(b) $\kappa'(G) \geq k - 1$.
Moreover, for each $k \geq 3$ there are infinitely many k-critical graphs with connectivity exactly 2.

Proof of (a): Suppose our k-critical G is disconnected and G_1 is a component of G. Since G is k-critical, its proper subgraphs G_1 and $G - V(G_1)$ have chromatic number at most $k - 1$. Let f_1 and $f - 2$ be their $(k - 1)$-colorings. Then $f_1 \cup f_2$ is a $(k - 1)$-coloring of G , a contradiction.

Theorem 5.6. Let $k \geq 3$ and G be a k-critical graph. Then
(a) $\kappa(G) \geq 2$;
(b) $\kappa'(G) \geq k - 1$.
Moreover, for each $k \geq 3$ there are infinitely many k-critical graphs with connectivity exactly 2.

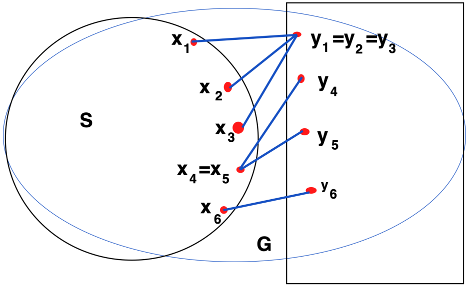
Proof of (a): Suppose our k-critical G is disconnected and G_1 is a component of G. Since G is k-critical, its proper subgraphs G_1 and $G - V(G_1)$ have chromatic number at most $k - 1$. Let f_1 and $f - 2$ be their $(k - 1)$-colorings. Then $f_1 \cup f_2$ is a $(k - 1)$-coloring of G , a contradiction.

Suppose now G is connected and v is a cut vertex in G. Let G' be a component of $G - v$, $G_1 = G - V(G')$ and $G_2 = G[V(G') + v]$. Again, since G is k-critical, for $i = 1, 2$, G_i has a $(k - 1)$-coloring f_i. We can rename the colors in f_2 to make $f_2(v) = f_1(v)$. Then again, $f_1 \cup f_2$ is a $(k - 1)$-coloring of G , a contradiction.

Proof of (b): Suppose $\kappa'(G) \leq k-2$. Then G has a vertex partition $(S, \overline{S})$ s.t. $E_G(S, \overline{S}) = \{x_i y_i : 1 \leq i \leq s\}$, where $s \leq k-2$, $\{x_1, \ldots, x_s\} \subset S$ and $\{y_1, \ldots, y_s\} \subset \overline{S}$.
Note that x_is do not need to be all distinct and y_is do not need to be all distinct.

Proof of (b): Suppose $\kappa'(G) \le k-2$. Then G has a vertex partition $(S, \overline{S})$ s.t. $E_G(S, \overline{S}) = \{x_i y_i : 1 \le i \le s\}$, where $s \le k-2$, $\{x_1, \ldots, x_s\} \subset S$ and $\{y_1, \ldots, y_s\} \subset \overline{S}$.
Note that x_is do not need to be all distinct and y_is do not need to be all distinct.

Let $G_1 = G[S]$ and $G_2 = G[\overline{S}]$. Since G is k-critical, for $i = 1, 2$, G_i has a $(k-1)$-coloring f_i.



We try to rename the colors of f_2 so that $f_1(x_i) \neq f_2(y_i)$ for all $1 \leq i \leq s$. There are $(k-1)!$ ways to rename these $k-1$ colors. Each of the edges $x_i y_i$ spoils the $(k-2)!$ cases where $f_1(x_i) = f_2(y_i)$.
Then the number of ways to rename the colors which are not spoiled is at least

$$(k-1)! - s((k-2)!) = (k-2)!((k-1)-s) \geq (k-2)! > 0.$$

Hence we can rename the colors in f_2 so that $f_1 \cup f_2$ is a $(k-1)$-coloring of G, a contradiction.

We try to rename the colors of f_2 so that $f_1(x_i) \neq f_2(y_i)$ for all $1 \leq i \leq s$. There are $(k-1)!$ ways to rename these $k-1$ colors. Each of the edges $x_i y_i$ spoils the $(k-2)!$ cases where $f_1(x_i) = f_2(y_i)$.
Then the number of ways to rename the colors which are not spoiled is at least

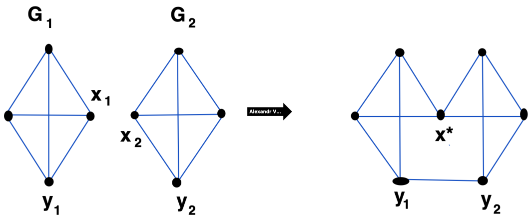
$$(k-1)! - s((k-2)!) = (k-2)!((k-1)-s) \geq (k-2)! > 0.$$

Hence we can rename the colors in f_2 so that $f_1 \cup f_2$ is a $(k-1)$-coloring of G, a contradiction.

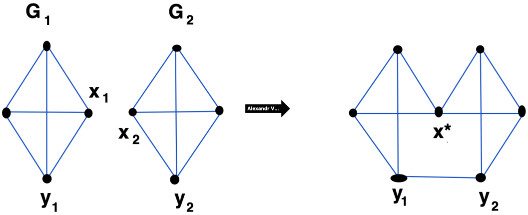
Proof of the "Moreover" part: We describe the Hájos Construction that creates from two k-critical graphs a new k-critical graph with connectivity exactly 2.

1) Take two disjoint k-critical graphs G_1 and G_2.
2) Choose an edge x_1y_1 in G_1 and an edge x_2y_2 in G_2.
3) Delete the edges x_1y_1 and x_2y_2, glue x_2 with x_1 into a new vertex x^*, add edge y_1y_2. Call new graph G^*.

1) Take two disjoint k-critical graphs G_1 and G_2.
2) Choose an edge x_1y_1 in G_1 and an edge x_2y_2 in G_2.
3) Delete the edges x_1y_1 and x_2y_2, glue x_2 with x_1 into a new vertex x^*, add edge y_1y_2. Call new graph G^*.



1) Take two disjoint k-critical graphs G_1 and G_2.
2) Choose an edge x_1y_1 in G_1 and an edge x_2y_2 in G_2.
3) Delete the edges x_1y_1 and x_2y_2, glue x_2 with x_1 into a new vertex x^*, add edge y_1y_2. Call new graph G^*.



By construction, set $\{x^*, y_1\}$ is separating in G^*. So $\kappa(G^*) = 2$.

Now we show that G^* is k-critical.

Suppose G^* has a $(k-1)$-coloring f. Since $f\mid_{V(G_1)}$ is NOT a $(k-1)$-coloring of G_1, $f(x^*) = f(y_1)$. Similarly, $f(x^*) = f(y_2)$. But then $f(y_1) = f(y_2)$, a contradiction.

Now we show that G^* is k-critical.
Suppose G^* has a $(k-1)$-coloring f. Since $f\mid_{V(G_1)}$ is NOT a $(k-1)$-coloring of G_1, $f(x^*) = f(y_1)$. Similarly, $f(x^*) = f(y_2)$. But then $f(y_1) = f(y_2)$, a contradiction.

Consider $G^* - y_1y_2$. Since G_1 and G_2 are k-critical, for $i = 1, 2$, $G_i - x_iy_i$ has a $(k-1)$-coloring f, and $f_i(y_i) = f_i(x_i)$. Then after permuting the colors in f_2 so that $f_2(x_2) = f_1(x_1)$, we get that $f_1 \cup f_2$ is a $(k-1)$-coloring of $G^* - y_1y_2$.

Now we show that G^* is k-critical.
Suppose G^* has a $(k-1)$-coloring f. Since $f\mid_{V(G_1)}$ is NOT a $(k-1)$-coloring of G_1, $f(x^*) = f(y_1)$. Similarly, $f(x^*) = f(y_2)$. But then $f(y_1) = f(y_2)$, a contradiction.

Consider $G^* - y_1y_2$. Since G_1 and G_2 are k-critical, for $i = 1, 2$, $G_i - x_iy_i$ has a $(k-1)$-coloring f, and $f_i(y_i) = f_i(x_i)$. Then after permuting the colors in f_2 so that $f_2(x_2) = f_1(x_1)$, we get that $f_1 \cup f_2$ is a $(k-1)$-coloring of $G^* - y_1y_2$.

Finally, let uv be any other edge of G^*. By symmetry, we may assume $\{u, v\} \subset V(G_1)$ (or one of them is x^*). Then $G_1 - uv$ has a $(k-1)$-coloring f_1 and $G_2 - x_2y_2$ has a $(k-1)$-coloring f_2. After permuting the colors in f_2 so that $f_2(x_2) = f_1(x_1)$, we get that $f_1 \cup f_2$ is a $(k-1)$-coloring of $G^* - uv$.