Graph coloring. Part 5

Lecture 36

Brooks' Theorem

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Proof. Fix $k \geq 3$. Suppose the theorem does not hold for this k. Choose a counter-example G with the smallest $|V(G)| + |E(G)|$. By the minimality, G is $(k+1)$-critical. So, by Theorem 5.6, G is 2-connected and k-regular. Let $n = |V(G)|$.

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Since G is not a complete graph, it has vertices v_1, v_2, v_3 such that $v_1v_2, v_2v_3 \in E(G)$ and $v_1v_3 \notin E(G)$.

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Since G is connected, we can cyclically rename the vertices of C as $y_j, y_{j+1}, \ldots, y_h, y_j$ so that $N(y_h) \subseteq V(C)$ and y_j has a neighbor y'_j outside of C.

Since G is $(k+1)$-critical, graph $G' = G - V(C)$ has a coloring f with colors $1, \ldots, k$. We may assume that $f(y'_j) = 1$.

We now extend f greedily to $V(C)$ using the order $y_h, y_{h-1}, \ldots, y_j$ of the vertices of C. In particular, $f(y_h) = 1$.

We claim that no more than k colors will be used: If $i > j$, then the neighbor y_{i-1} of y_i is not colored, and hence y_i has at most $k - 1$ forbidden colors. If $i = j$, then y_i has two neighbors, y_h and y_j', of the same color. This contradicts the choice of G.

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Case 2: $h = n$, i.e. $P = x_1, \ldots, x_n$. Since $k \geq 3$, x_2 has a neighbor x_j for some $j \geq 4$. Consider a greedy coloring of G w.r.t. order

$$x_1, x_3, x_4, \ldots, x_{j-1}, x_n, x_{n-1}, \ldots, x_j, x_2.$$

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On the other hand, x_2 has two neighbors, x_1 and x_3, of the same color. This proves the theorem.

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Proof: Consider a partition $V(G) = V_1 \cup V_2$ of $V(G)$

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Then by Brooks' Theorem, $\chi(G_1) \leq 3$ and $\chi(G_2) \leq 3$. Hence, $\chi(G) \leq 3 + 3 = 6$.

A (proper) k-edge-coloring of a graph G is a mapping $f : E(G) \to \{1, \ldots, k\}$ such that

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Observations: 1. If G has a loop, then it has no k-edge-coloring for any k.
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We call $f^{-1}(i)$ a color class of f. By definition, a k-edge-coloring of a graph G is a partition of $E(G)$ into k matchings.

The edge chromatic number, $\chi'(G)$, of a graph G is the minimum positive integer k s.t. G has a k-edge-coloring. Sometimes it is called the chromatic index of G.
G is k-edge-colorable if $\chi'(G) \leq k$.

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Examples: 1. Complete graphs. 2. Cycles. 3. Bipartite graphs.
4. Petersen graph.
5. 3-regular graphs with a cut edge.

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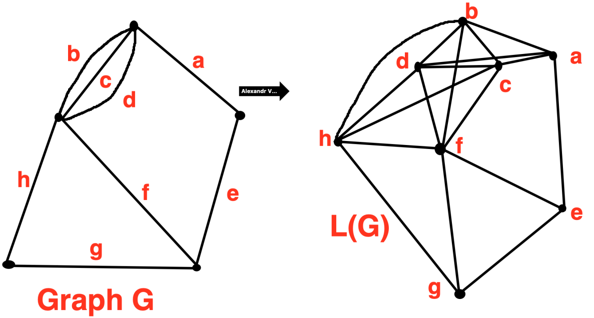

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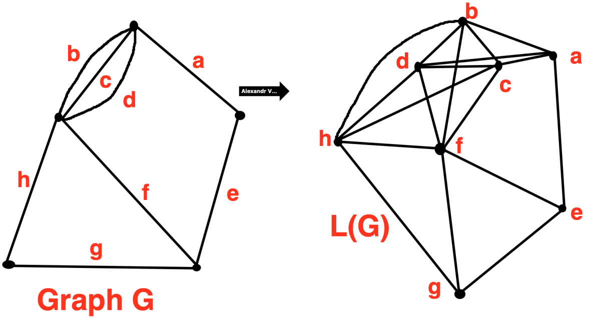

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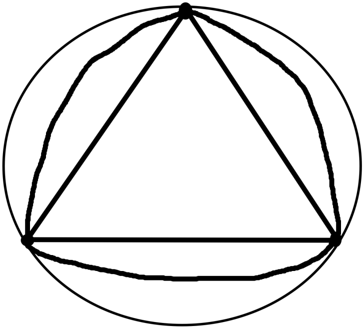
Shannon's application and example.



Graph S_6

$\Delta(S_k) = k$ and $\chi'(S_k) = \left\lfloor \frac{3k}{2} \right\rfloor$.