Eulerian circuits

Lecture 4

Lemma 1.2 (Lemma 1.2.15 in the book) : Every closed walk of odd length contains an odd cycle.

Proof. We use induction on the length ℓ of the closed walk W. If $\ell = 1$, then W is a loop, i.e., a cycle of length 1.
Now, assume $\ell > 1$. Consider W starting and ending at a vertex u. If W does not contain any repeated vertices (except starting and ending at u), then W is a cycle. So we assume that W contains a repeated vertex v.

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Consider W as a walk starting at v, and note that we can split W into two shorter closed walks, W' and W'', both starting at v. Since W had an odd number of edges, either W' or W'' also has an odd number of edges. So by induction, it contains an odd cycle, which is also contained in W. □

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Let G_1 be a nontrivial component of G. By (1), G_1 has a cycle, say C_0. Let G_2 be obtained from G by deleting all edges of C_0.

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Hence $E(G)$ can be partitioned into edge sets of cycles $C_1, \ldots, C_t$ and C_0. $\square$

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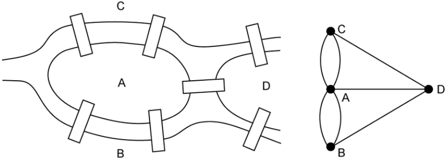

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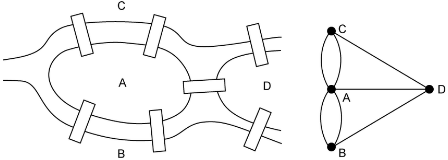

Recall that for the existence of an Eulerian circuit in G it is necessary that degree of every vertex in G is even.

Another necessary condition is that G has at most one nontrivial component (isolated vertices do not count).

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Let C be a longest circuit in G. If C contains every edge, this is an Eulerian circuit and we are done.

If not, consider the subgraph $G' \subseteq G$ with $V(G') = V(G)$ and $E(G') = E(G) \setminus E(C)$ (i.e. we delete all the edges in C from G). Since every vertex has even degree in C (by virtue of C being a circuit) and even degree in G, every vertex also has even degree in G'. Since G is connected, there must be a vertex v that has positive degree in both G' and C.

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By Lemma 1.5(2), G' has a cycle C' containing v. Then we can concatenate C and C' to get a longer circuit in G, contradicting maximality of C. □

We can now derive a similar result for Eulerian trails.

Corollary 1.7: Given a graph G and distinct vertices u and v in it, G has an Eulerian u, v-trail if and only if

(a) degree of every vertex in $V(G) - \{u, v\}$ is even, degrees of u and v are odd, and

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Now, Theorem 1.6 for G' implies our corollary for G. □

Extremal problems on graphs

By extremal problems on graphs we mean problems when we ask for either the minimum or the maximum number of edges in n-vertex graphs with given properties.
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Proof. The fact that $f(n) \geq \left\lfloor \frac{n^2}{4} \right\rfloor$ follows from the example of the complete bipartite graph $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$, since it is bipartite (and thus has no 3-cycles) and has exactly $\left\lfloor \frac{n^2}{4} \right\rfloor$ edges.