Mantel's Theorem, degree sequences

Lecture 5

Extremal problems on graphs

By extremal problems on graphs we mean problems when we ask for either the minimum or the maximum number of edges in n-vertex graphs with given properties.
A very typical extremal problem was resolved by Mantel in 1907: he determined the maximum number of edges in an n-vertex simple graph with no triangles.

Theorem 1.8 (Theorem 1.3.22) **Mantel's Theorem, 1907**: Let $f(n)$ be the maximum number of edges in a simple n-vertex graph with no triangles. For each $n \geq 1$, $f(n) = \left\lfloor \frac{n^2}{4} \right\rfloor$.

Proof. The fact that $f(n) \geq \left\lfloor \frac{n^2}{4} \right\rfloor$ follows from the example of the complete bipartite graph $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$, since it is bipartite (and thus has no 3-cycles) and has exactly $\left\lfloor \frac{n^2}{4} \right\rfloor$ edges.

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Suppose now that (1) holds for all $n \leq k-1$ and let G be a simple k-vertex graph with no 3-cycles and $|E(G)| = f(k)$. Let xy be any fixed edge in G. Since G has no 3-cycles, every $z \in V(G) - x - y$ is adjacent to at most one of x and y. Thus

$$\text{the number of edges incident to } x \text{ or } y \text{ is at most } k-1. \tag{2}$$

Let $G' = G - x - y$. By construction, G' is a simple $(k-2)$-vertex triangle-free graph. By the induction assumption, $|E(G')| \leq f(k-2) = \left\lfloor \frac{(k-2)^2}{4} \right\rfloor$. But then by (2),

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$$|E(G)| \leq |E(G')| + k - 1 \leq \left\lfloor \frac{(k-2)^2}{4} \right\rfloor + k - 1$$

$$= \left\lfloor \frac{(k-2)^2 + 4(k-1)}{4} \right\rfloor = \left\lfloor \frac{k^2}{4} \right\rfloor,$$

as claimed. This proves the induction step and thus the theorem. □

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An example of an incorrect proof.

Degree sequences

Proposition 1.9: A sequence $\mathbf{d} = (d_1, \ldots, d_n)$ of nonnegative integers is the degree sequence of a graph if and only if $\sum_{i=1}^{n} d_i$ is even.

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Theorem 1.10 (Theorem 1.3.31) **Havel–Hakimi**: The only graphic sequence of length 1 is (0). For $n > 1$ a sequence $\mathbf{d} = (d_1, \ldots, d_n)$ of integers with $d_1 \geq d_2 \geq \ldots \geq d_n \geq 0$ is graphic if and only if the sequence $\mathbf{d}' = (d_2 - 1, d_3 - 1, \ldots, d_{d_1+1} - 1, d_{d_1+2}, \ldots, d_n)$ is graphic.

Example: $(5, 5, 3, 3, 2, 2, 1, 1)$.

Proof. ($\Longleftarrow$) Suppose $\mathbf{d}'$ is graphic. Let G' be a simple graph with degree sequence $\mathbf{d}'$ and vertex set $\{v_2, \ldots, v_n\}$ where $d_{G'}(v_i) = d'_{i-1}$.
Let G be the graph obtained by adding to G' a new vertex v_1 adjacent to $v_2, \ldots, v_{d_1+1}$. Then the degree sequence of G is $\mathbf{d}$. Thus $\mathbf{d}$ is graphic.

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($\Longrightarrow$) Suppose $\mathbf{d}$ is graphic. Among the simple graphs with degree sequence $\mathbf{d}$ and vertex set $V = \{v_1, v_2, \ldots, v_n\}$ where the degree of v_i is d_i for all i, choose a graph G in which

$$v_1 \text{ has the most neighbors in } S = \{v_2, \ldots, v_{d_1+1}\}. \tag{3}$$

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$$v_1 \text{ has the most neighbors in } S = \{v_2, \ldots, v_{d_1+1}\}. \qquad (3)$$

If $N_G(v_1) = S$, then the degree sequence of $G - v_1$ is $\mathbf{d}'$, and hence $\mathbf{d}'$ is graphic. Thus assume v_1 is not adjacent to some $v_i \in S$.

In this case, v_1 has a neighbor $v_j \notin S$. Since $i < j$, $d_i \geq d_j$. Moreover, v_i is not adjacent to v_1 while v_j is. Together with $d_i \geq d_j$, this yields that there is $v_k \in V$ adjacent to v_i but not to v_j.

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Then the graph G_1 obtained from G by deleting edges $v_1 v_j$ and $v_i v_k$ and adding edges $v_1 v_i$ and $v_j v_k$ is a simple graph with the same degree sequence as G.
But in this graph, v_1 has more neighbors in S, contradicting (3). □