01/31/24

Math 412 **HW3**

Due Wednesday, February 7, 2024

Solve four of the next five problems.

1. Extending the proof of Mantel's Theorem given in class (see lecture slides), prove that for each $n \geq 1$, the only n-vertex triangle-free simple graph with the maximum number of edges is $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$. (Other proofs do not count.)

2. Using Problem 1, find the minimum number of edges that may have a simple n-vertex connected graph with independence number at most two. (Hint: consider the complement.)

3. Given an integer $n \geq 3$ and a nonincreasing list $\mathbf{d} = (d_1, \ldots, d_n)$ of nonnegative integers, let $\mathbf{d}'(n)$ be obtained from $\mathbf{d}$ by deleting d_n and subtracting 1 from d_n largest elements remaining in the list. Prove that $\mathbf{d}$ is graphic if and only if $\mathbf{d}'(n)$ is graphic. (Hint: Mimic the proof of Havel–Hakimi Theorem.)

4. Let G be a digraph such that
(a) $d^+(v) = d^-(v)$ for all but two special vertices x and y; and
(b) $d^+(x) - d^-(x) = d^-(y) - d^+(y) = 5$.
Prove that G has 5 edge-disjoint paths from x to y. (Hint: Use properties of Eulerian digraphs.)

5. For every odd n, construct an n-vertex tournament in which every vertex is a king. Does there exist such a tournament with 4 vertices?

Problems below review basic concepts and their ideas could be used in the tests.

WARMUP PROBLEMS: Section 1.3: # 1, 2, 4, 5, 8, 9, 12. Section 1.4: # 1, 3. Do not write these up!

OTHER INTERESTING PROBLEMS: Section 1.3: # 18, 24, 32, 40, 41, 57, 63. Section 1.4: # 21, 26, 28, 37. Do not write these up!