Spanning trees, III and matchings

Lecture 12

A lemma

Lemma 2.7 : Let G be a connected loopless graph with weighted edges, where $w(e) \geq 0$ for every $e \in E(G)$.
Let $T_1, \ldots, T_k$ be vertex-disjoint trees contained in G such that $V(T_1) \cup \ldots \cup V(T_k) = V(G)$.
Let e_0 be an edge of the minimum weight among the edges of G connecting $V(T_1)$ with $V(G) - V(T_1)$.
Then among the containing $E(T_1) \cup \ldots \cup E(T_k)$ spanning trees of G of minimum weight, there is a tree containing e_0.

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Proof. Let $n = V(G)$. Let T_0 be a spanning tree of G containing $E(T_1) \cup \ldots \cup E(T_k)$ of minimum weight.

Suppose $e_0 = xy$ where $x \in V(T_1)$ and $y \in V(G) - V(T_1)$.
If $e_0 \in E(T_0)$, then we are done.

Otherwise, $T' = T_0 + e_0$ is a connected graph with n edges containing exactly one cycle, say C. By construction, $e_0 \in E(C)$.
Since $x \in V(T_1)$ and $y \in V(G) - V(T_1)$, cycle C contains another edge e_1 connecting $V(T_1)$ with $V(G) - V(T_1)$.
Then $T'' := T' - e_1$ is a connected graph with $n - 1$ edges; hence a spanning tree of G. Moreover, by the choice of e_0, $w(e_0) \leq w(e_1)$.

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Therefore, $\sum_{e \in E(T'')} w(e) \leq \sum_{e \in E(T_0)} w(e)$. It follows that T'' also is a spanning tree of G containing $E(T_1) \cup \ldots \cup E(T_k)$ of minimum weight. □

Main theorems in Chapter 2:

1. A Characterization Theorem for trees (Theorem 2.2).
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2. Jordan's Theorem on centers of trees (Theorem 2.3).
3. Theorem on Prüfer codes, Cayley's Formula.
4. Matrix Tree Theorem (Theorem 2.6).
5. Prim's and Kruskal's algorithms.

Matchings

A matching in a graph is a set of non-loop edges that are pairwise disjoint.
The **size** of a matching is the number of edges in it.

In particular, an empty set of edges is a matching (of size 0).
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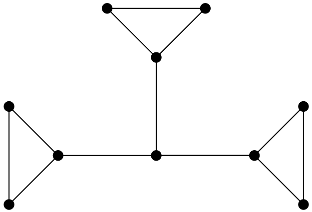
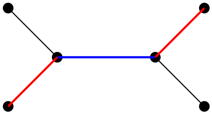
A matching is **perfect** in a graph G if it covers all vertices of G.



The main problem is to find a matching in a graph G with the most edges.
A **maximal matching** in a graph G is a matching that is not a subset of any larger matching.

A **maximum matching** is a matching that has the most edges over all matchings of G.



The size of a maximum matching in G is denoted by $\alpha'(G)$.

Recall that the independence number of G is denoted by $\alpha(G)$.

Given a matching M in a graph G, an **M-alternating path** in G is a path that alternates between edges in M and not in M.

An **M-augmenting** path is an M-alternating path whose endpoints are not in any edge of M.
Since an M-augmenting path must start and end with an edge that is not in M, any M-augmenting path is of odd length, and has more edges outside M than in M.

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Proof. (A) $\Rightarrow$ (B) (We prove ($\neg$B) $\Rightarrow$ ($\neg$A)). If P is an M-augmenting path, then by removing from M the edges in $M \cap E(P)$ and adding the edges in $E(P) - M$, we obtain a matching larger than M.

(B) $\Rightarrow$ (A) (We prove ($\neg$A) $\Rightarrow$ ($\neg$M)). Suppose there is a matching M' with $|M'| > |M|$. Consider the graph G' with vertex set $V(G)$ and edge set $M \cup M'$.

Since the edges set of G' is the union of two matchings, $\Delta(G') \leq 2$, each component of G' is a path or a cycle of even length. Each cycle or even-length path in G' is made up of the same number of edges from M and M'.

Since $|M'| > |M|$, there is a path P with more edges in M' than in M. The only way to have it is that the first and last edges of P are in $M' - M$. Then the endpoints of P are not covered by M. This means P is an M-augmenting path. □