Matchings in bipartite graphs

Lecture 13

Matchings

A matching in a graph is a set of non-loop edges that are pairwise disjoint.
The size of a maximum matching in G is denoted by $\alpha'(G)$.

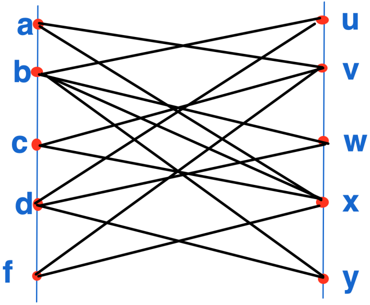
Given a matching M in a graph G, an ***M*-alternating path** in G is a path that alternates between edges in M and not in M.

An ***M*-augmenting** path is an M-alternating path whose endpoints are not in any edge of M.

Theorem 3.1 (Berge)
(A) A matching M in a graph G is maximum
if and only if (B) G does not contain any M-augmenting path.

Bipartite graphs

Given a bipartite graph $G = (X, Y; E)$, certainly, $\alpha'(G) \le \min\{|X|, |Y|\}$. But it can be smaller.



The fundamental result for bipartite graphs is the Hall Theorem.

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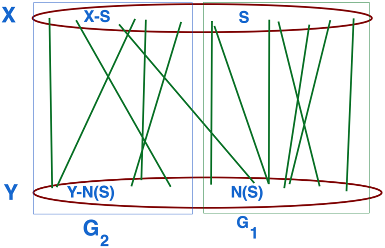
Induction Step. Suppose the theorem is true for all graphs with less than m edges. Let $G = (X, Y; E)$ have m edges.

Case 1: $|N(S)| = |S|$ for some $\emptyset \neq S \subsetneq X$. Define induced subgraphs G_1 and G_2 of G: $V(G_1) = S \cup N_G(S)$ and $G_2 = G - V(G_1)$.



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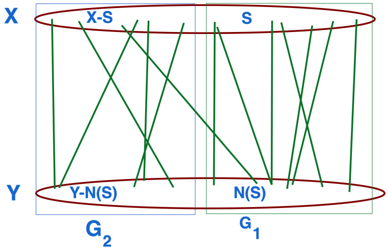

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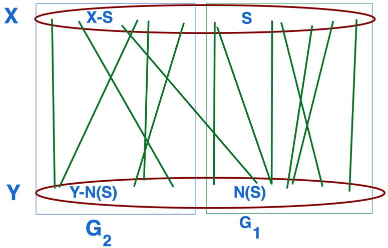

Claim 1. (1) holds for G_1.

Claim 2. (1) holds for G_2.

Indeed, if there is $T \subset X - S$ with $|N_{G_2}(T)| < |T|$, then

$$|N_G(S \cup T)| = |N_G(S)| + |N_{G_2}(T)| < |S| + |T| = |S \cup T|,$$

a contradiction.

In view of Claims 1 and 2, by the induction assumption, G_1 has a matching M_1 covering S and G_2 has a matching M_2 covering $X - S$. Now $M_1 \cup M_2$ covers X.

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Case 2:

$$|N_G(S)| \geq |S| + 1 \qquad \forall \emptyset \neq S \subsetneq X. \tag{2}$$

Choose any $x_0 \in X$. Since $d(x_0) \geq 1$, there is $y_0 \in N(x_0)$. Let $G' = G - x_0 - y_0$.

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By (2), for each $\emptyset \neq S \subset X - x_0$,

$$|N_{G'}(S)| \geq |N_G(S)| - 1 \geq (|S| + 1) - 1 = |S|.$$

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So (1) holds for G', and by IH, G' has a matching M' covering $X - x_0$.

Then matching $M' \cup \{x_0 y_0\}$ covers X, as claimed. □

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Proof. Let $B = (X, Y; E)$ be a k-regular bipartite graph. Since each edge of B has exactly one endpoint in X, and exactly one in Y,

$$|E(B)| = \sum_{v \in X} d(v) = k|X|,$$

and

$$|E(B)| = \sum_{v \in Y} d(v) = k|Y|,$$

so $|X| = |Y|$.
Thus each matching that covers X is perfect. Let us check that Hall's condition is satisfied.

Let $S \subseteq X$. There are exactly $k|S|$ edges incident with vertices in S, so there are at least $k|S|$ edges incident with $N(S)$, and the total number of edges incident with $N(S)$ is $k|N(S)|$, so

$$k|S| \leq k|N(S)|,$$

which is equivalent to Hall's condition. Thus, we are done by Hall's Theorem. □

Systems of distinct representatives.

Vertex covers

A vertex cover of a graph G is a set S of vertices in G such that each edge of G has at least one end in S.

Trivially, $V(G)$ is a vertex cover of G. The problem is to find a vertex cover of the minimum cardinality.

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Observation C: For each graph G, $\alpha'(G) \leq \beta(G) \leq 2\alpha'(G)$.

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Proof. Let $G = (X, Y; E)$ be a bipartite graph with parts X and Y. By Observation C, we need only to prove $\alpha'(G) \geq \beta(G)$.

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Claim: (i) $\forall A \subseteq Q \cap X, \quad |N(A) - Q \cap Y| \geq |A|$.
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By the claim and Hall's Theorem, graph $G[(Q \cap X) \cup (Y - Q)]$ has a matching M_X covering $Q \cap X$ and graph $G[(Q \cap Y) \cup (X - Q)]$ has a matching M_Y covering $Q \cap Y$.