Covers in bipartite graphs and stable matchings

Lecture 15

Hall's Theorem and vertex covers

Theorem 3.2 (P. Hall): An X, Y-bigraph G has a matching covering X if and only if

$$|N(S)| \geq |S| \qquad \forall S \subseteq X. \tag{1}$$

A vertex cover of a graph G is a set S of vertices in G such that each edge of G has at least one end in S.

Observation C: For each graph G, $\alpha'(G) \leq \beta(G) \leq 2\alpha'(G)$.

Theorem 3.4 (König, Egerváry, 1931): For each bipartite graph G,

$$\alpha'(G) = \beta(G). \tag{2}$$

An edge cover of a graph G is a set T of edges in G such that each vertex of G is an end of at least one edge in T.

Trivially, if G has isolated vertices, then it has no edge cover. If G has no isolated vertices, then $E(G)$ is an edge cover of G. The problem is to find an edge cover of the minimum cardinality.

The minimum cardinality of an edge cover of G is denoted by $\beta'(G)$.

Theorem 3.5 (Gallai, 1959): For each n-vertex graph G with no isolated vertices, $\alpha'(G) + \beta'(G) = n$.

Proof of Theorem 3.5. Let G be an n-vertex graph G with no isolated vertices.

Part 1: We prove $\alpha'(G) + \beta'(G) \le n$. Let M be a matching in G with $|M| = \alpha'(G)$. It does not cover exactly $n - 2\alpha'(G)$ vertices. Each of these vertices we can cover with a special edge. Thus

$$\beta'(G) \le \alpha'(G) + (n - 2\alpha'(G)) = n - \alpha'(G),$$

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Let L be an edge cover of G with $|L| = \beta'(G)$. Consider the subgraph G_L of G spanned by the edges in L.

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Let L be an edge cover of G with $|L| = \beta'(G)$. Consider the subgraph G_L of G spanned by the edges in L.

By the minimality of L, G_L does not contain cycles and paths of length 3. Thus G_L is a star forest.

Let k be the number of components in G_L. Then G_L has a matching M with k edges.
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On the other hand, $|L| = n - k$.

Therefore,

$$\beta'(G) + \alpha'(G) \geq |L| + |M| \geq (n-k) + k = n,$$

as claimed. $\square$

Corollary: For each bipartite graph G with no isolated vertices, $\alpha(G) = \beta'(G)$.

The stable matching problem

In a famous paper "College admissions and the stability of marriage" from 1962, Gale and Shapley (awarded the Nobel Prize for this in 2012) considered the following problem.

There are n men and n women. Each man has his own linear order of preferences among women and each woman has her own linear order of preferences among men.

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An example:

$$\begin{array}{ccc} & \text{Men}\,[w,x,y,z] & \\ w: & c > b > a > d \\ x: & a > b > c > d \\ y: & a > c > b > d \\ z: & c > b > a > d \end{array} \qquad \begin{array}{cc} & \text{Women}\,[a,b,c,d] \\ a: & z > x > y > w \\ b: & y > w > x > z \\ c: & w > x > y > z \\ d: & x > y > z > w. \end{array}$$

There are $n!$ ways to marry all men to all women. Each such way corresponds to a perfect matching in $K_{n,n}$ with parts X and Y, where X is the set of the men and Y is the set of the women.

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An *unstable pair* is such a matching M is a pair (x, y) with $x \in X$ and $y \in Y$ such that x is not married to y but likes y more than his wife and y likes x more than her husband.

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Comment: The language is about marriage, but this setting models also admissions of students or graduate students to colleges. Applicants could be viewed as men and universities as groups of women. If the number of vacancies is less than the number of applicants, we can add extra women to whom nobody wants to marry.

Gale-Shapley Proposal Algorithm

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Iteration: Each man proposes to the woman highest in his list among those who had not rejected him, yet.
If each woman receives exactly one proposal, then Stop and output this matching.
Otherwise, each woman says ”Maybe” to the highest in her list proposer and rejects other proposers. Each man deletes the woman rejecting him from his list. Go to the next iteration.

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Theorem 3.6 (Gale and Shapley, 1962): The above algorithm produces a stable matching.

Proof of Theorem 3.6

Observation 1: If a woman rejects somebody at least once, then she has a proposer till the very end, and the position of each next ”Maybe” man in her list can only grow.

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Observation 3: The algorithm stops after at most n^2 rounds and produces some perfect matching M.

Observation 4: The produced matching M is stable.

Indeed, suppose M is not stable. Then there is $x \in X$ and $a \in Y$ such that a is higher in the list of x than $M(x)$ and x is higher in the list of a than $M(a)$.
This means x proposed to a at some step(s), and at some Step j, a rejected him, because of a better proposer. But then by Observation 1, $M(a)$ is higher in her list than x. $\square$