Stable matchings
and
matchings in general graphs

Lecture 16

An example:

Men $[w, x, y, z]$	Women $[a, b, c, d]$
$w: c > b > a > d$	$a: z > x > y > w$
$x: a > b > c > d$	$b: y > w > x > z$
$y: a > c > b > d$	$c: w > x > y > z$
$z: c > b > a > d$	$d: x > y > z > w.$

An *unstable pair* is such a matching M is a pair (x, y) with $x \in X$ and $y \in Y$ such that x is not married to y but likes y more than his wife and y likes x more than her husband.

A perfect matching in such $K_{n,n}$ with preference list is **stable**, if it has no unstable pairs.

Gale-Shapley Proposal Algorithm

Input: Preference rankings of men and women.

Goal: Find a stable matching.

Iteration: Each man proposes to the woman highest in his list among those who had not rejected him, yet.
If each woman receives exactly one proposal, then Stop and output this matching.
Otherwise, each woman says ”Maybe” to the highest in her list proposer and rejects other proposers. Each man deletes the woman rejecting him from his list. Go to the next iteration.

Theorem 3.6 (Gale and Shapley, 1962): The above algorithm produces a stable matching.

Proof of Theorem 3.6

Observation 1: If a woman rejects somebody at least once, then she has a proposer till the very end, and the position of each next ”Maybe” man in her list can only grow.

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Observation 4: The produced matching M is stable.

Indeed, suppose M is not stable. Then there is $x \in X$ and $a \in Y$ such that a is higher in the list of x than $M(x)$ and x is higher in the list of a than $M(a)$.
This means x proposed to a at some step(s), and at some Step j, a rejected him, because of a better proposer. But then by Observation 1, $M(a)$ is higher in her list than x. □

An example

Men $[u, v, x, y, z]$	Women $[a, b, c, d, e]$
$u: \; c > b > e > d > a$	$a: \; y > x > u > v > z$
$v: \; c > d > e > a > b$	$b: \; u > x > v > z > y$
$x: \; a > b > c > d > e$	$c: \; z > x > y > u > v$
$y: \; a > e > d > b > c$	$d: \; v > x > u > z > y$
$z: \; c > e > b > a > d$	$e: \; v > u > y > x > z.$

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Moreover, EVERY WOMAN gets the WORST husband she can get in a stable matching.

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This algorithm is used to assign the graduates of American medical schools as residents at hospitals over the country.

Matchings in general graphs

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An odd component of a graph G is a component with an odd number of vertices, and $o(G)$ denotes the number odd components of G

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For example, if G has an odd component, then G has no p.m. Similarly, if G has a set S of vertices s.t. $G - S$ has more than $|S|$ odd components, then again G has no p.m.

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The importance of the above observation follows from the famous Tutte's Theorem.

Theorem 3.7 (Tutte, 1947): A graph G has a p.m. if and only if

$$o(G-S) \le |S| \qquad \forall S \subseteq V(G). \tag{1}$$

Proof. The ”only if” part is easy. We prove the ”if” part.
If this part does not hold for n-vertex simple graphs, then there is an n-vertex simple graph G satisfying (1) and no p.m. with the most edges.

Since adding an edge to G preserves (1), any such adding leads to a graph with a p.m.
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Case 1: All components of $G - U$ are complete graphs. Since by (1), $o(G - U) \leq |U|$, we construct a p.m. by hand.

Case 2: Some component G' of $G - U$ is not a complete graph. Then G' contains an induced path P_3, say with vertices x, y and z (in this order).

Since $y \notin U$, there is $w \in V(G) - N(y)$.

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Let $e_1 = xz$, $e_2 = wy$. Let $G_i = G + e_i$. Then for $i = 1, 2$, G_i contains a p.m. M_i.
Furthermore, $e_1 \in M_1 - M_2$ and $e_2 \in M_2 - M_1$.

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Consider the graph F with edge set $M_1 \cup M_2$. By definition, $\Delta(F) = 2$, and every component is either an edge (belonging to both, M_1 and M_2) or a cycle whose edges alternately belong to M_1 and M_2.

Let C_i be the cycle in F containing e_i.

Case 2.1: $C_2 \neq C_1$. Then we define a new perfect matching in F:
$M = (M_1 - (M_1 \cap E(C_1))) \cup (M_2 \cap E(C_1))$.

This M contains neither e_1 nor e_2. Hence it is a p.m. in G, a contradiction.

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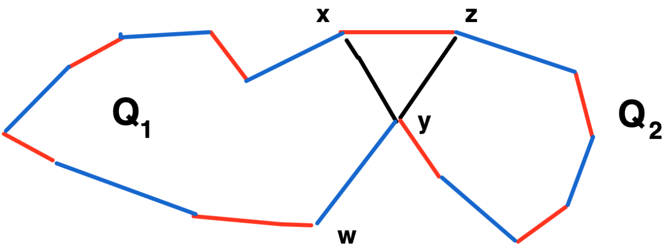
Case 2.2: $C_2 = C_1$.



Let Q_1 be the x, y-path in $C_1 - xz$ and Q_2 be the z, y-path in $C_1 - xz$

By symmetry, we may assume that $wy \in E(Q_1)$. Let M_0 be obtained from M_1 by deleting the edges in $M_1 \cap Q_2$ and adding edge xy and all edges in $M_2 \cap Q_2$.

Then M_0 covers all vertices of G and contains neither e_1 nor e_2. This contradiction proves the theorem. □