Matchings in general graphs

Lecture 17

For $k \geq 1$, a k-factor in a graph G is a spanning k-regular subgraph of G. So, a 1-factor is simply a p.m.

Theorem 3.7 (Tutte, 1947): A graph G has a p.m. if and only if

$$o(G - S) \leq |S| \qquad \forall S \subseteq V(G). \tag{1}$$

Proof. The "only if" part is easy. We prove the "if" part.
If this part does not hold for n-vertex simple graphs, then there is an n-vertex simple graph G satisfying (1) and no p.m. with the most edges.

Since adding an edge to G preserves (1), any such adding leads to a graph with a p.m.
Let U be the set of vertices in G of degree $n - 1$.

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Let U be the set of vertices in G of degree $n - 1$.

Case 1: All components of $G - U$ are complete graphs. Since by (1), $o(G - U) \leq |U|$, we construct a p.m. by hand.

Case 2: Some component G' of $G - U$ is not a complete graph. Then G' contains an induced path P_3, say with vertices x, y and z (in this order).

Since $y \notin U$, there is $w \in V(G) - N(y)$.

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Let $e_1 = xz$, $e_2 = wy$. Let $G_i = G + e_i$. Then for $i = 1, 2$, G_i contains a p.m. M_i.
Furthermore, $e_1 \in M_1 - M_2$ and $e_2 \in M_2 - M_1$.

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Consider the graph F with edge set $M_1 \cup M_2$. By definition, $\Delta(F) = 2$, and every component is either an edge (belonging to both, M_1 and M_2) or a cycle whose edges alternately belong to M_1 and M_2.

Let C_i be the cycle in F containing e_i.

Case 2.1: $C_2 \neq C_1$. Then we define a new perfect matching in F:
$M = (M_1 - (M_1 \cap E(C_1))) \cup (M_2 \cap E(C_1))$.

This M contains neither e_1 nor e_2. Hence it is a p.m. in G, a contradiction.

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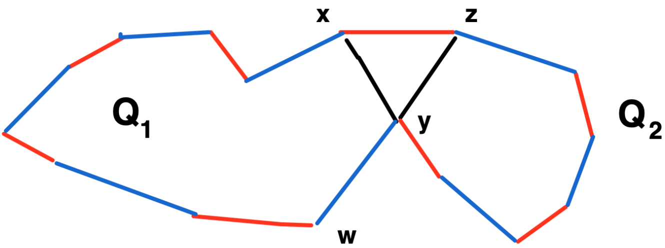
Case 2.2: $C_2 = C_1$.



Let Q_1 be the x, y-path in $C_1 - xz$ and Q_2 be the z, y-path in $C_1 - xz$

By symmetry, we may assume that $wy \in E(Q_1)$. Let M_0 be obtained from M_1 by deleting the edges in $M_1 \cap Q_2$ and adding edge xy and all edges in $M_2 \cap Q_2$.

Then M_0 covers all vertices of G and contains neither e_1 nor e_2. This contradiction proves the theorem. $\square$

Corollary 3.8 (Petersen, 1891): Every 3-regular graph with no cut-edges has a p.m.

Proof. Suppose a 3-regular graph G with no cut-edges has no p.m.
Then by Theorem 3.7, there is $S \subseteq V(G)$ s.t. $o(G - S) > |S|$.

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Suppose $S = \{v_1, \ldots, v_s\}$ and odd components of $G - S$ are $H_1, \ldots, H_t$, where $t \geq s + 1$. We claim that for each $1 \leq j \leq t$,

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the number of edges between H_j and S is odd. $(*)$

Indeed, $\sum_{v \in V(H_j)} d(v) = 3|V(H_j)|$ and hence is odd. Every edge inside H_j contributes 2 to $\sum_{v \in V(H_j)} d(v)$, and each edge between S and H_j contributes 1. This proves $(*)$.

Since G has no cut edges, by $(*)$ for each $1 \leq j \leq t$,

the number of edges between H_j and S is at least 3. $(**)$

By $(**)$,

$$|E(S, V(G) - S)| \geq 3t.$$

On the other hand,

$$|E(S, V(G) - S)| \leq \sum_{w \in S} d(w) = 3s < 3t.$$

This contradiction proves the corollary. $\square$

Theorem 3.9 (Petersen, 1891): For every $k \geq 1$, every $2k$-regular graph has a 2-factor.

Proof. It is enough to prove the theorem for connected graphs. So, suppose G is a connected $2k$-regular graph with vertex set $V = \{v_1, \ldots, v_n\}$. Then G has an Eulerian circuit C. Let $e_1, \ldots, e_m$ be the (directed) edges of C.

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We construct an auxiliary bigraph H as follows. The parts of H are V and $V' = \{v'_1, \ldots, v'_n\}$.
For every e_j in C, if e_j leads from v_i to v_h, we add edge $v_i v'_h$ to $E(H)$.

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So by Marriage Theorem, H has a p.m. M.

The edges of M form a 2-factor in G. □