Connectivity, II

Lecture 21

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Proof. Let $n \geq 3$.
$(\Leftarrow)$ We prove the contrapositive. Suppose an n-vertex G is not 2-connected. Since $n \geq 3$, by Lemma 4.1 there is an $x \in V(G)$ such that $G - x$ is disconnected. This means there is a partition $V(G) = \{x\} \cup A \cup B$ with $A \neq \emptyset$ and $B \neq \emptyset$ such that no edge connects A with B.

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Let $a \in A$ and $b \in B$. Then each a, b-path in G contains x. Thus G has no internally disjoint a, b-paths.

($\Rightarrow$) Let G be 2-connected. We use induction on $d(u, v)$.

Base of induction: $d(u, v) = 1$. Since $\kappa'(G) \geq \kappa(G) \geq 2$, $G - uv$ is connected; thus it contains a u, v-path P. Another u, v-path is uv.

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Induction Step. Suppose the theorem holds for all pairs of vertices at distance at most $k - 1$. Take any two vertices u and v s.t. $d(u, v) = k$. Let $P = v_0 v_1 \ldots v_k$ be a shortest path from $v_0 = u$ to $v_k = v$.

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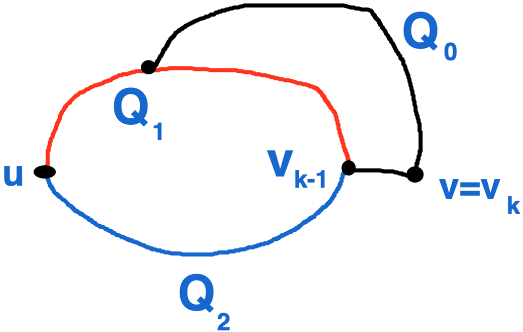
Then $d(u, v_{k-1}) = k - 1 < k$. So by induction, there are internally disjoint u, v_{k-1}-paths Q_1 and Q_2. Note that $Q_1 \cup Q_2$ is a cycle.

Case 1: $v \in V(Q_1 \cup Q_2)$. Then on the cycle $Q_1 \cup Q_2$ we find internally disjoint u, v-paths.

Case 2: $v \notin V(Q_1 \cup Q_2)$. Since $\kappa(G) \geq 2$, $G - v_{k-1}$ has a path Q_0 from v to $V(Q_1 \cup Q_2) - v_{k-1}$, see below.



Using paths Q_0, Q_1, Q_2 and edge $v_{k-1}v$, we easily find two internally disjoint u, v-paths. □

Lemma 4.5 (Expansion Lemma): Let G be k-connected and G' be obtained from G by adding a new vertex y adjacent to at least k vertices in G. Then G' is k-connected.

Proof. Since G is k-connected, $|V(G)| \geq k + 1$.
Assume G' is not k-connected. Then there is a separating set $S \subset V(G')$ with $|S| \leq k - 1$.

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Case 2: $y \notin S$. Let A be the vertex set of the component of $G' - S$ containing y and $B = V(G') - A - S$. If $|A| \geq 2$, then S is a separating set in G and $|S| \leq k-1$, a contradiction.

So assume $A = \{y\}$. Then $S \supseteq N_{G'}(y)$, but $|N_{G'}(y)| \geq k$, a contradiction. □

A characterization theorem

Theorem 4.6 (Characterization theorem of 2-connected graphs): Let G be a graph with $|V(G)| \geq 3$. The following conditions are equivalent:

(A) G is connected and has no cut vertices.

(B) $\forall x, y \in V(G)$, there are internally disjoint x, y-paths.

(C) $\forall x, y \in V(G)$, there is a cycle containing both x and y.

(D) $\delta(G) \geq 1$ and $\forall e, e' \in E(G)$, there is a cycle containing both e and e'.

(F) $\delta(G) \geq 2$ and $\forall e, e' \in E(G)$, there is a cycle containing both e and e'.

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(F) $\delta(G) \geq 2$ and $\forall e, e' \in E(G)$, there is a cycle containing both e and e'.

Proof. Theorem 4.4 proves $(A) \Leftrightarrow (B)$.
Clearly, $(B) \Leftrightarrow (C)$ and $(F) \Rightarrow (D)$.

To show (D) $\Rightarrow$ (C), we prove ($\neg$C) $\Rightarrow$ ($\neg$D)).

The negation of (C) means that there are vertices x and y not in a common cycle. If (D) holds, there is an edge e incident to x and an edge e' incident to y. Hence there is no cycle containing e and e'.

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To finish the theorem we need only to show (A) $\Rightarrow$ (F).

Suppose G is connected and has no cut vertices. Then $\delta(G) \geq 2$. Now take any two edges, $e = xy$ and $e' = uv$ (possibily, $x = u$). Let G' by obtained from G by adding a new vertex a adjacent to x and y and a new vertex b adjacent to u and v. By the Expansion Lemma, G' is 2-connected.

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By Whitney's Theorem, G' has a cycle C containing a and b. Then C must use edges xa, ay, ub and bv. Replacing these four edges with edges e and e', we obtain a cycle in G containing e and e'. $\square$

A subdivision of an edge e connecting vertices u and v in a graph G is the operation of replacing edge e with a path u, w, v through a new vertex w.

Corollary 4.7. If G is 2-connected , then the graph G' obtained by subdividing an edge of G also is 2-connected .

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Proof. Let G' be obtained from G by subdividing an edge e connecting vertices u and v with vertex w. Let $e_1 = uw$ and $e_2 = wv$.

We will prove that G' satisfies conditions (F) in Theorem 4.6.

Clearly, $\delta(G') = 2$. To prove that (F) holds for G', consider two arbitrary edges g and h.

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Case 1: $\{g, h\} \cap \{e_1, e_2\} = \emptyset$. Since G is 2-connected , it contains a cycle C containing g and h. If $e \notin E(C)$, then C is a cycle in G' containing g and h.

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Case 2: $|\{g, h\} \cap \{e_1, e_2\}| = 1$, say $g = e_1$ and $h \neq e_2$. Again, since G is 2-connected , it contains a cycle C containing e and h.
Then the cycle C' obtained from C by replacing e with e_1 and e_2 is a cycle in G' containing g and h.

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Case 3: $\{g, h\} = \{e_1, e_2\}$. Again, G contains a cycle C containing e. Again, the cycle C' obtained from C by replacing e with e_1 and e_2 is a cycle in G' containing g and h. □

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An ear decomposition of a graph G is a partition $(P_0, P_1, \ldots, P_k)$ of the edge set of G s.t.
(a) P_0 is a cycle of length at least 3, and
(b) for $i = 1, \ldots, k$, P_i is an ear of $P_0 \cup P_1 \cup \ldots \cup P_i$.