Menger's Theorem

Lecture 22

A characterization theorem

Theorem 4.6 (Characterization theorem of 2-connected graphs): Let G be a graph with $|V(G)| \geq 3$. The following conditions are equivalent:

(A) G is connected and has no cut vertices.

(B) $\forall x, y \in V(G)$, there are internally disjoint x, y-paths.

(C) $\forall x, y \in V(G)$, there is a cycle containing both x and y.

(D) $\delta(G) \geq 1$ and $\forall e, e' \in E(G)$, there is a cycle containing both e and e'.

(F) $\delta(G) \geq 2$ and $\forall e, e' \in E(G)$, there is a cycle containing both e and e'.

Corollary 4.7. If G is 2-connected , then the graph G' obtained by subdividing an edge of G also is 2-connected .

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(b) for $i = 1, \ldots, k$, P_i is an ear of $P_0 \cup P_1 \cup \ldots \cup P_i$.

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Theorem 4.8. A graph G is 2-connected if and only if G has an ear decomposition. Moreover, if G is 2-connected, then every cycle in G of length at least 3 is the initial cycle in some ear decomposition of G.

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Theorem 4.8. A graph G is 2-connected if and only if G has an ear decomposition. Moreover, if G is 2-connected, then every cycle in G of length at least 3 is the initial cycle in some ear decomposition of G.

Proof. ($\Leftarrow$) Let $(P_0, P_1, \ldots, P_k)$ be an ear decomposition of a graph G.

We prove the stronger statement that for each $0 \leq i \leq k$, $P_0 \cup P_1 \cup \ldots \cup P_i$ forms a 2-connected graph.

This is true for $i = 0$ because every cycle of length at least 3 is 2-connected.

For induction step, observe that $P_0 \cup P_1 \cup \ldots \cup P_i$ is obtained by adding path P_i to the 2-connected graph $P_0 \cup P_1 \cup \ldots \cup P_{i-1}$.

Note that adding a path can be considered as first adding an edge, and then a sequence of subdivisions.

By Corollary 4.7 and the fact that adding an edge to a 2-connected graph results in a 2-connected graph, $P_0 \cup P_1 \cup \ldots \cup P_i$ is a 2-connected graph.

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By Corollary 4.7 and the fact that adding an edge to a 2-connected graph results in a 2-connected graph, $P_0 \cup P_1 \cup \ldots \cup P_i$ is a 2-connected graph.

($\Rightarrow$) Let G be a 2-connected graph and C be a cycle in G of length at least 3.
We let $G_0 = C$ and try to construct $G_1, G_2, \ldots$ so that for each $i \geq 1$, G_i is obtained from G_{i-1} by adding a path whose end vertices are in $V(G_{i-1})$, but internal vertices are not.

Suppose G_{i-1} is constructed. If $G_{i-1} = G$, then we are done. Suppose not. Then there exists an edge $e \in E(G) - E(G_{i-1})$ s.t. at least one end of e is in $V(G_{i-1})$, say the ends of e are u and v, and $u \in V(G_{i-1})$.

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Let e' be an edge in $E(G_{i-1})$ incident to u. By Part (F) of Theorem 4.6, G has a cycle C containing e and e'.

Let P be the path in C starting from u, containing v and ending at the first after u vertex of C that is in $V(G_{i-1})$.

Then internal vertices of P are not in $V(G_{i-1})$, so we let G_i be obtained from G_{i-1} by adding P. □

Main version of Menger's Theorem

Let G be a graph or a digraph and $x, y \in V(G)$ with $xy \notin E(G)$. Then an x, y-cut is a set $S \subset V(G) - \{x, y\}$ such that $G - S$ has no x, y-paths.

Define $\kappa_G(x, y)$ be the minimum size of an x, y-cut in G.

Also, by $\lambda_G(x, y)$ denote the maximum number of internally disjoint x, y-paths in G.

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Clearly, $\kappa_G(x, y) \geq \lambda_G(x, y)$.

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Theorem 4.9 (Menger): Let G be a graph, $x, y \in V(G)$ and $xy \notin E(G)$. Then $\kappa_G(x, y) = \lambda_G(x, y)$.

Proof. Assume the theorem does not hold. Then there is a counterexample, i.e. a graph G and two vertices $x, y \in V(G)$ with $xy \notin E(G)$ such that

$$\kappa_G(x, y) > \lambda_G(x, y) \tag{1}$$

with the minimum $|V(G)|$. Let $n = |V(G)|$.

Proof setup

By the minimality of $|V(G)|$, for each H with $|V(H)| < n$

$$\kappa_H(u, v) = \lambda_H(u, v) \text{ for each } u, v \in V(H) \text{ with } uv \notin E(H). \quad (2)$$

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Let $k = \kappa_G(x,y)$. If $k = 0$, then also $\lambda_G(x,y) = 0$, and the theorem holds. So assume $k \geq 1$.

Since $N(x)$ and $N(y)$ are x,y-cuts,

$$k \leq \min\{|N(x)|, |N(y)|\}. \quad (3)$$

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In a series of claims below, we derive more and more

properties of G. Eventually, we will show that it does not exist.

Claims and conclusion

Claim 1. Every x, y-cut with k vertices is $N(x)$ or $N(y)$.

Claim 2. $V(G) = \{x, y\} \cup N(x) \cup N(y)$.

Claim 3. $N(x) \cap N(y) = \emptyset$.

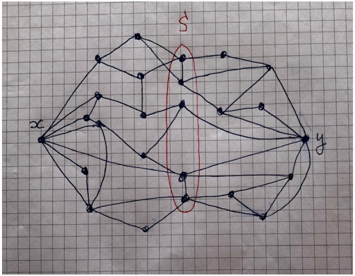
Claim 4. $|N(x)| = |N(y)| = k$.

Claim 5. Let H be the bipartite graph with parts $N(x)$ and $N(y)$ such that $u \in N(x)$ is adjacent to $v \in N(y)$ iff $uv \in E(G)$. Then H has a perfect matching.

Since a perfect matching in H corresponds to k internally disjoint x, y-paths in G, Claim 5 yields that $\lambda_G(x, y) \geq k$, a contradiction to (1).

Proof of Claim 1.

Suppose G has an x, y-cut S with k vertices distinct from $N(x)$ or $N(y)$.



Let G' be the component of $G - S$ containing x. Let G_x be obtained from $G - G'$ by adding the new vertex x' adjacent to all vertices of S. Graph G_y is defined symmetrically, but instead of $G - G'$ it uses $G[S \cup V(G')]$.

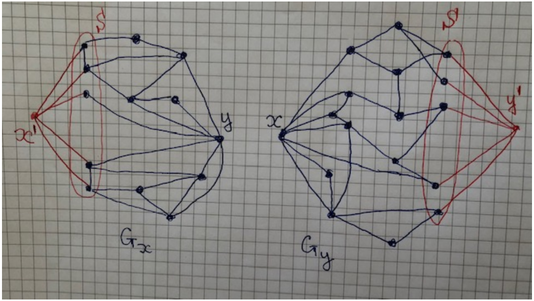
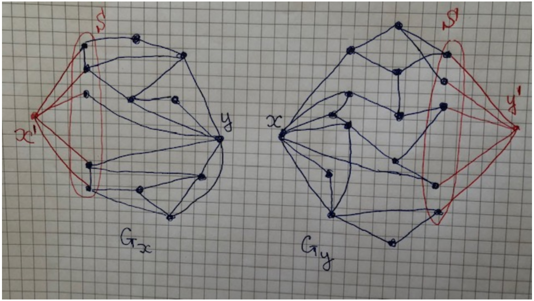


Figure: Graphs G_x and G_y.

Since S does not contain $N(x)$ or $N(y)$, each of G_x and G_y is smaller than G.



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Since S does not contain $N(x)$ or $N(y)$, each of G_x and G_y is smaller than G.

Any x', y-cut S' in G_x is also an x, y-cut in G. It follows that $\kappa_{G_x}(x', y) \geq k$. In view of S, it is exactly k. So by the minimality of G, $\lambda_{G_x}(x', y) = k$.

Similarly, $\lambda_{G_y}(x, y') = k$.

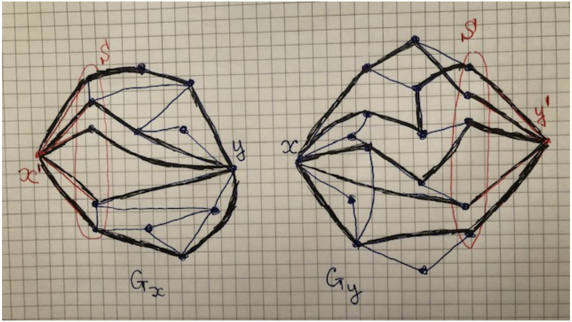
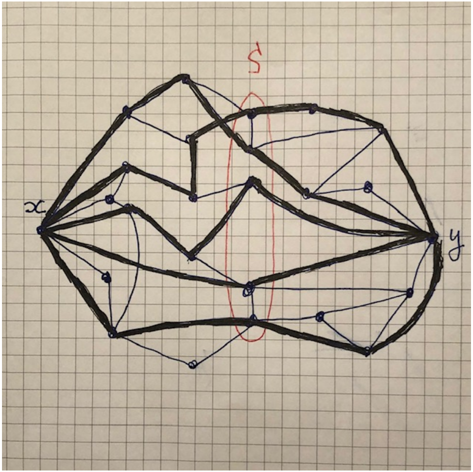


Figure: Paths in graphs G_x and G_y.

Let $P_1, \ldots, P_k$ be int.-disjoint x', y-paths in G_x and $Q_1, \ldots, Q_k$ be int.-disjoint x, y'-paths in G_y.

Then for every $1 \leq i \leq k$, $R_i = (Q_i - y') \cup (P_i - x')$ is an x, y-path in G. Also, all $R_1, \ldots, R_k$ are int.-disjoint, contradicting (1).



This proves Claim 1.

Proof of Claim 2: $V(G) = \{x, y\} \cup N(x) \cup N(y)$.

Suppose G has a vertex $z \in V(G) - (\{x, y\} \cup N(x) \cup N(y))$. By Claim 1, z does not belong to any x, y-cut of size k. This means that for the graph $G' = G - z$

$$\kappa_{G'}(x, y) = k.$$

By the minimality of G,

$$\lambda_{G'}(x, y) = \kappa_{G'}(x, y) = k.$$

So,

$$\lambda_G(x, y) \geq \lambda_{G'}(x, y) = k,$$

contradicting (1).

Proof of Claim 3: $N(x) \cap N(y) = \emptyset$.

Suppose G has a vertex $u \in N(x) \cap N(y)$.

Let $G' = G - u$. Then $\kappa_{G'}(x, y) \geq k - 1$.

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By the minimality of G, $\lambda_{G'}(x, y) = \kappa_{G'}(x, y) \geq k - 1$. Let $P_1, \ldots, P_{k-1}$ be int.-disjoint x, y-paths in G'.
Adding to them path $P_k = x, u, y$ we obtain k int.-disjoint x, y-paths in G. This contradicts (1).

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Suppose $|N(x)| \geq k + 1$ and $v \in N(x)$. Let $G' = G - v$.

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Suppose $|N(x)| \geq k + 1$ and $v \in N(x)$. Let $G' = G - v$. By Claim 1, v is not in any x, y-cut of size k.

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Hence $\kappa_{G'}(x, y) = k$.
By the minimality of G, $\lambda_{G'}(x, y) = \kappa_{G'}(x, y) = k$. So,

$$\lambda_G(x, y) \geq \lambda_{G'}(x, y) = k,$$

contradicting (1).

Proof of Claim 5: The auxiliary bigraph H has a perfect matching.

Recall that H is the bipartite graph obtained from G by deleting x and y and all edges inside $N(x)$ and $N(y)$. Also recall that by Claim 4, $|N(x)| = |N(y)| = k$.

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But by the choice of A,

$$|S| = |N(x) - A| + |N_H(A)| = k - |A| + |N_H(A)| < k,$$

contradicting the definition of k.

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This proves Theorem 4.9.