Flows in networks, II

Lecture 26

Lemma 4.15. Every positive circulation f in a network G can be represented as the sum of at most $|E(G)| - 1$ positive flows along cycles.

Theorem 4.16. Every positive flow f in a network G can be represented as the sum of at most $|E(G)|$ positive flows along cycles, along s, t-paths and along t, s-paths.

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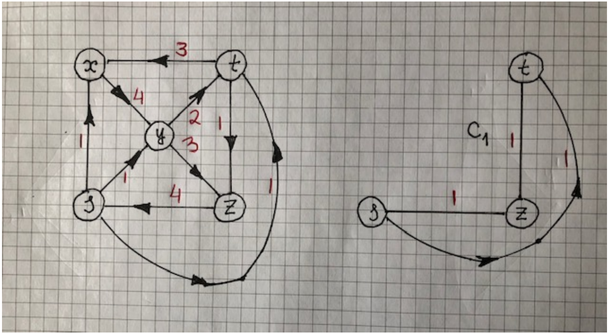
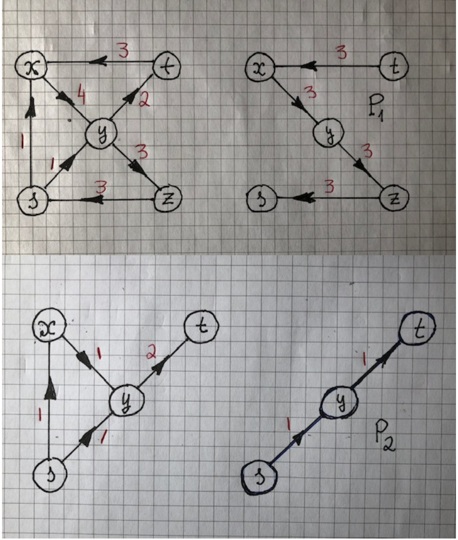
An example:





Hence $f = \phi(C_1, 1) + \phi(P_1, 3) + \phi(P_2, 1) + \phi(P_3, 1)$.

A function $f : E \rightarrow \mathbf{R}$ is called a flow in G if for every vertex $v \in V - s - t$,

$$div_f(v) = \sum_{e \in E^+(v)} f(e) - \sum_{e \in E^-(v)} f(e) = 0. \qquad (1)$$

If $0 \leq f(e) \leq \mathbf{c}(e)$ for every $e \in E$, then the flow is called feasible (for G).

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An (s, t)-cut in a network $G = (V, E, s, t, \{\mathbf{c}(e)\}_{e \in E})$ is a partition $(S, \overline{S})$ of V into sets S and $\overline{S}$ such that $s \in S$ and $t \in \overline{S}$.

The capacity of $(S, \overline{S})$ is

$$\mathbf{c}(S, \overline{S}) = \sum_{xy \in E : x \in S, y \in \overline{S}} \mathbf{c}(xy). \qquad (2)$$

Important inequality

Claim 4.17. For every feasible flow f in in a network $G = (V, E, s, t, \{\mathbf{c}(e)\}_{e \in E})$ and every s, t-cut $(S, \overline{S})$ of V,

$$M(f) \leq \mathbf{c}(S, \overline{S}). \tag{3}$$

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Proof. Consider $F(f, S) = \sum_{v \in S} div_f(v)$.
By the definition of a flow, since $s \in S$ and $t \notin S$,
$F(f, S) = div_f(s) = M(f)$.

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On the other hand, by (1),

$$F(f, S) = \sum_{v \in S} \left(\sum_{e \in E^+(v)} f(e) - \sum_{e \in E^-(v)} f(e) \right).$$

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Let $e = xy$. Then e contributes $f(e)$ to $div_f(x)$ and $-f(e)$ to $div_f(y)$.
So, if $\{x, y\} \subset S$, then the net contribution of e is $f(e) - f(e) = 0$.
Also, if $\{x, y\} \subset \overline{S}$, then the net contribution of e is 0.
If $x \in S$ and $y \in \overline{S}$, then e contributes $f(e)$ into the RHS of (4).
Finally, if $y \in S$ and $x \in \overline{S}$, then e contributes $-f(e)$.

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Hence $F(f,S) = \sum_{xy\in E: x\in S, y\in\overline{S}} f(xy) - \sum_{yx\in E: x\in S, y\in\overline{S}} f(yx)$.

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Since $0 \le f(e) \le \mathbf{c}(e)$ for each $e \in E$, the last sum is at most

$$\sum_{xy\in E: x\in S, y\in\overline{S}} \mathbf{c}(xy) = \mathbf{c}(S,\overline{S}).$$

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Since $0 \le f(e) \le \mathbf{c}(e)$ for each $e \in E$, the last sum is at most $\sum_{xy \in E: x \in S, y \in \overline{S}} \mathbf{c}(xy) = \mathbf{c}(S, \overline{S})$. So, $M(f) \le \mathbf{c}(S, \overline{S})$.