Graph coloring. Part 4

Lecture 35

For $k \geq 1$, a graph G is k-critical, if $\chi(G) = k$, but for each proper subgraph G' of G,

$$\chi(G') \leq k - 1.$$

Theorem 5.6. Let $k \geq 3$ and G be a k-critical graph. Then
(a) $\kappa(G) \geq 2$;
(b) $\kappa'(G) \geq k - 1$.
Moreover, for each $k \geq 3$ there are infinitely many k-critical graphs with connectivity exactly 2.

Proof of the "Moreover" part: We describe the Hájos Construction that creates from two k-critical graphs a new k-critical graph with connectivity exactly 2.

1) Take two disjoint k-critical graphs G_1 and G_2.
2) Choose an edge x_1y_1 in G_1 and an edge x_2y_2 in G_2.
3) Delete the edges x_1y_1 and x_2y_2, glue x_2 with x_1 into a new vertex x^*, add edge y_1y_2. Call new graph G^*.

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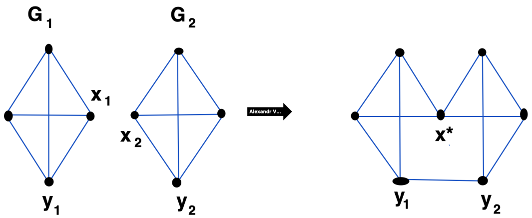

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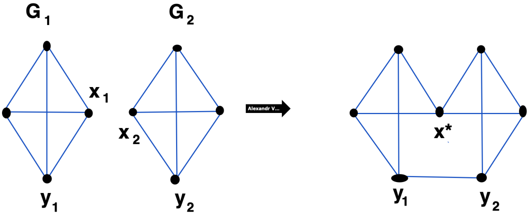

By construction, set $\{x^*, y_1\}$ is separating in G^*. So $\kappa(G^*) = 2$.

Now we show that G^* is k-critical.
Suppose G^* has a $(k-1)$-coloring f. Since $f\mid_{V(G_1)}$ is NOT a $(k-1)$-coloring of G_1, $f(x^*) = f(y_1)$. Similarly, $f(x^*) = f(y_2)$. But then $f(y_1) = f(y_2)$, a contradiction.

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Consider $G^* - y_1y_2$. Since G_1 and G_2 are k-critical, for $i = 1, 2$, $G_i - x_iy_i$ has a $(k-1)$-coloring f, and $f_i(y_i) = f_i(x_i)$. Then after permuting the colors in f_2 so that $f_2(x_2) = f_1(x_1)$, we get that $f_1 \cup f_2$ is a $(k-1)$-coloring of $G^* - y_1y_2$.

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Finally, let uv be any other edge of G^*. By symmetry, we may assume $\{u, v\} \subset V(G_1)$ (or one of them is x^*). Then $G_1 - uv$ has a $(k-1)$-coloring f_1 and $G_2 - x_2y_2$ has a $(k-1)$-coloring f_2. After permuting the colors in f_2 so that $f_2(x_2) = f_1(x_1)$, we get that $f_1 \cup f_2$ is a $(k-1)$-coloring of $G^* - uv$.

Definitions

Mycielski's Construction: $M_3 = C_5$. Suppose M_k is a triangle-free graph with $\chi(M_k) = k$ and $V(M_k) = V_k = \{v_1, ..., v_{n_k}\}$. Let $V'_k = \{u_1, ..., u_{n_k}\}$. Then $V(M_{k+1}) = V_k \cup V'_k \cup \{w\}$, $M_{k+1}[V_k] = M_k$, $N_{M_{k+1}}(w) = V'_k$ and for each $1 \leq j \leq n_k$, $N_{M_{k+1}}(u_j) = N_{M_k}(v_j) \cup \{w\}$.

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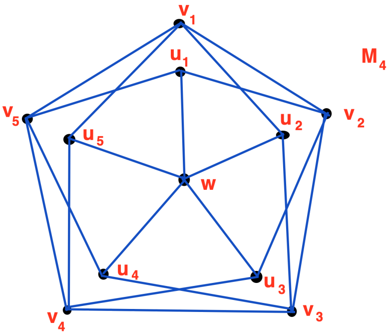

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Then color k_0 is not used on V'_{k_0}. Let $W = \{v_{i_1}, \ldots, v_{i_s}\}$ be the set of the vertices in V_{k_0} colored with k_0. We will recolor them: for each $1 \leq j \leq s$, recolor v_{i_j} with $f(u_{i_j})$. Then color k_0 is not used in the new coloring f' of M_k.

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We claim that

$$f'(v_i) \neq f'(v_j) \quad \text{for each edge } v_i v_j \in E(M_{k_0}). \qquad (1)$$

Indeed, suppose $f'(v_i) = f'(v_j)$. If $f(v_i) \neq k_0$ and $f(v_j) \neq k_0$, then the colors of v_i and v_j did not change, but $f(v_i) \neq f(v_j)$, a contradiction. If $f(v_i) = k_0 = f(v_j)$, then v_i and v_j cannot be adjacent.

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This shows that the difference $\chi(G) - \omega(G)$ and the ratio $\frac{\chi(G)}{\omega(G)}$ can be arbitrarily large.

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Since G is not a complete graph, it has vertices v_1, v_2, v_3 such that $v_1v_2, v_2v_3 \in E(G)$ and $v_1v_3 \notin E(G)$.

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Since G is connected, we can cyclically rename the vertices of C as $y_j, y_{j+1}, \ldots, y_h, y_j$ so that $N(y_h) \subseteq V(C)$ and y_j has a neighbor y_j' outside of C.

Since G is $(k+1)$-critical, graph $G' = G - V(C)$ has a coloring f with colors $1, \ldots, k$. We may assume that $f(y_j') = 1$.

We now extend f greedily to $V(C)$ using the order $y_h, y_{h-1}, \ldots, y_j$ of the vertices of C. In particular, $f(y_h) = 1$.

We claim that no more than k colors will be used: If $i > j$, then the neighbor y_{i-1} of y_i is not colored, and hence y_i has at most $k - 1$ forbidden colors. If $i = j$, then y_i has two neighbors, y_h and y_j', of the same color. This contradicts the choice of G.

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Then by Brooks' Theorem, $\chi(G_1) \leq 3$ and $\chi(G_2) \leq 3$. Hence, $\chi(G) \leq 3 + 3 = 6$.