Directed graphs

Lecture 6

Revisiting graphic degree sequences

Proof. ($\Longrightarrow$) Suppose $\mathbf{d}$ is graphic. Among the simple graphs with degree sequence $\mathbf{d}$ and vertex set $V = \{v_1, v_2, \ldots, v_n\}$ where the degree of v_i is d_i for all i, choose a graph G in which

$$v_1 \text{ has the most neighbors in } S = \{v_2, \ldots, v_{d_1+1}\}. \qquad (1)$$

If $N_G(v_1) = S$, then the degree sequence of $G - v_1$ is $\mathbf{d}'$, and so $\mathbf{d}'$ is graphic. Thus assume v_1 is not adjacent to some $v_i \in S$.

In this case, v_1 has a neighbor $v_j \notin S$. Since $i < j$, $d_i \geq d_j$. And v_i is not adjacent to v_1 while v_j is. Together with $d_i \geq d_j$, this yields that there is $v_k \in V$ adjacent to v_i but not to v_j.

Then the graph G_1 obtained from G by deleting edges $v_1 v_j$ and $v_i v_k$ and adding edges $v_1 v_i$ and $v_j v_k$ is a simple graph with the same degree sequence as G.
But in this graph, v_1 has more neighbors in S, contradicting (1).

Directed graphs

Graphs are good to model symmetric binary relations, but often we need to model antisymmetric relations.

A directed graph (a digraph) is a pair consisting of a vertex set $V = V(G)$, an edge set $E = E(G)$ and a relation associating with each $e \in E(G)$ two vertices (not necessarily distinct) called its tail and head.

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Out-neighbors and in-neighbors, degrees, outdegrees and indegrees of vertices in digraphs. Simple digraphs.

Adjacency and incidence matrices for digraphs: definitions and examples.

Proposition 1.11 (**Degree Sum Formula for digraphs**): For every digraph G $\sum_{v \in V(G)} d^+(v) = \sum_{v \in V(G)} d^-(v) = |E(G)|$.

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Paths, trails, walks and cycles in digraphs: Definitions and examples.

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The **underlying graph** of a digraph D is the graph G s.t. $V(G) = V(D)$ and every directed edge of D constitutes a unique (undirected) edge of G.

Connected and weakly connected digraphs.

Theorem 1.12 (Theorem 1.4.24) **Euler's Theorem for digraphs**: A digraph G has an Eulerian circuit if and only if
(a) $d^+(v) = d^-(v)$ for every vertex v in G and
(b) G has at most one nontrivial weak component.

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Proof of Theorem 1.12 now practically repeats the proof of Theorem 1.6 with Lemma 1.13 replacing Lemma 1.5: choose a largest circuit in G, and if does not contain all edges, then we are able to enlarge it.

de Bruijn graphs

The vertices of the de Bruijn graph B_n are the n-dimensional $0, 1$-vectors.
And B_n has an edge from $(\alpha_1, \dots, \alpha_n)$ to $(\beta_1, \dots, \beta_n)$ if and only if

$$\alpha_2 = \beta_1,\ \alpha_3 = \beta_2, \dots, \alpha_n = \beta_{n-1}.$$

de Bruijn graphs