Directed graphs

Lecture 7

Connected and weakly connected digraphs.

Theorem 1.12 (Theorem 1.4.24) **Euler's Theorem for digraphs**: A digraph G has an Eulerian circuit if and only if
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To prove it, we can use an analog of Lemma 1.5:
Lemma 1.13: If $d^+(v) = d^-(v)$ for every vertex v in G, then we can partition $E(G)$ into (directed) cycles.

Proof. As in Lemma 1.5, consider the longest (directed) paths in G and use induction on the number of edges. □

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Proof of Theorem 1.12 now practically repeats the proof of Theorem 1.6 with Lemma 1.13 replacing Lemma 1.5: choose a largest circuit in G, and if does not contain all edges, then we are able to enlarge it.

de Bruijn graphs

The vertices of the de Bruijn graph B_n are the n-dimensional $0, 1$-vectors.
And B_n has an edge from $(\alpha_1, \ldots, \alpha_n)$ to $(\beta_1, \ldots, \beta_n)$ if and only if
$\alpha_2 = \beta_1$, $\alpha_3 = \beta_2, \ldots, \alpha_n = \beta_{n-1}$.

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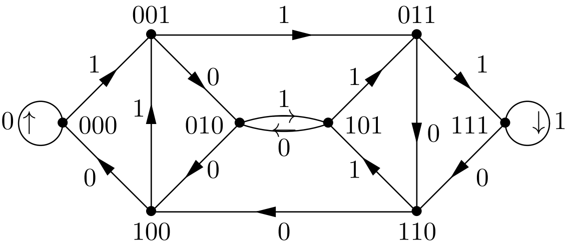

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Every Eulerian circuit in B_{n-1} yields such a cyclic arrangement.

Kings in tournaments

A vertex v in a digraph D is a king if every vertex in D can be reached from v by a (directed) path of length at most 2.

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Proof. Let x be a vertex of maximum out-degree in a tournament T.

Note that $V(T) = \{x\} \cup N^+(x) \cup N^-(x)$. If x is not a king, then there should be $y \in V(T)$ not reachable from x in at most two steps. Such y must be in $N^-(x)$. Fix this y.

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For each $z \in N^+(x)$, $yz \in E(T)$ since otherwise (x, z, y) would be our path. But then $N^+(x) \subset N^+(y)$ and $d^+(x) < d^+(y)$, contradicting the choice of x. □

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5. Havel-Hakimi Theorem on graphic sequences.

Trees

A graph with no cycle is called **acyclic**.
A **tree** is a connected acyclic graph.

So, an acyclic graph is also called a **forest.**
By definition, each component of a forest is a tree.

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Let G be a connected graph and v be a leaf in G. Let $G' = G - v$.
Since G is connected, for any vertices $u, w \in V(G) - v$, there is a u, w-path $P(u, w)$. It does not contain v, since every internal vertex of $P(u, w)$ has degree at least 2.

Therefore, $P(u, w)$ is in G'. Since each $P(u, w)$ is in G', graph G' is connected. □