Trees

Lecture 8

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5. Havel-Hakimi Theorem on graphic sequences.

Trees

A graph with no cycle is called **acyclic**.
A **tree** is a connected acyclic graph.

So, an acyclic graph is also called a **forest.**
By definition, each component of a forest is a tree.

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Let G be a connected graph and v be a leaf in G. Let $G' = G - v$.
Since G is connected, for any vertices $u, w \in V(G) - v$, there is a u, w-path $P(u, w)$. It does not contain v, since every internal vertex of $P(u, w)$ has degree at least 2.

Therefore, $P(u, w)$ is in G'. Since each $P(u, w)$ is in G', graph G' is connected. □

Characterization of trees

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Proof. (A) $\Rightarrow$ (B,C). We use induction on n. For $n = 1$ the claim is obvious. Suppose that $n > 1$ and every tree with $k < n$ vertices has exactly $k - 1$ edges. Let G be any n-vertex tree. By Lemma 2.1 (a), G has a leaf, say v. By Lemma 2.1 (b), $G - v$ has $(n - 1) - 1$ edges. But then G has $n - 1$ edges, as claimed.

(B) $\Rightarrow$ (A,C). Suppose G is connected and has $n-1$ edges. Deleting an edge from a cycle in G leaves it connected. Do this procedure until the final graph G' has no cycles but is connected. By definition, G' is a tree. Since (A) $\Rightarrow$ (B), G' has $n-1$ edges. But then $G' = G$.

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(C) $\Rightarrow$ (A,B). Suppose G has no cycles and has $n-1$ edges. Let $G_1, \ldots, G_k$ be the components of G. Let n_i (respectively, e_i) be the number of vertices (respectively, edges) in G_i. By construction, each G_i is a tree. Since (A) $\Rightarrow$ (B), for each $1 \leq i \leq k$, $e_i = n_i - 1$. Then

$$n - 1 = |E(G)| = \sum_{i=1}^{k} e_i = \sum_{i=1}^{k} (n_i - 1) = n - k.$$

Thus, $k = 1$, as claimed.

(A) $\Rightarrow$ (D) (We prove ($\neg$D) $\Rightarrow$ ($\neg$A)). If (D) does not hold, then there are $u, v \in V(G)$ s.t. either (a) there are no u, v-paths or (b) there are more than one u, v-paths.
If (a) holds, then G is disconnected, and if (b) holds, then G has a cycle. In any case, G is not a tree.

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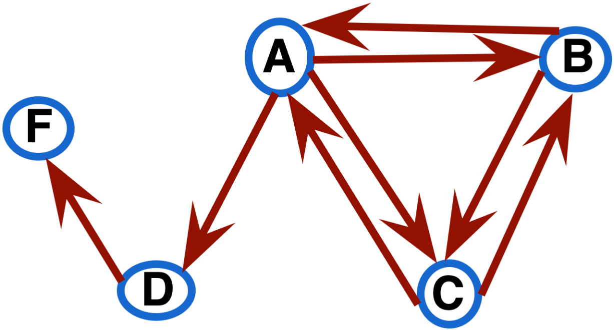

(D) $\Rightarrow$ (F) Suppose (D) holds for G and $u, v \in V(G)$. Since by (D), G has exactly one u, v-path, $G + uv$ will have exactly one cycle (passing through uv).

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(F) $\Rightarrow$ (A) (We prove ($\neg$A) $\Rightarrow$ ($\neg$F)). If (A) does not hold, then either (a) G has a cycle, say C, or (b) G is disconnected.
If (a) holds, then adding an edge with both ends on C creates at least one more cycle, so (F) does not hold.
If (a) does not hold but (b) holds, then adding an edge with ends in distinct components would create a graph with no cycles, violating (F) again. $\square$

Distances in graphs

Let G be a graph and let $u, v \in V(G)$.
If u and v are in the same component of G, then the **distance** from u to v is the length of the shortest u, v-path in G, and we write $d_G(u, v)$ for this (or often just $d(u, v)$). If u and v are in different components, then we define $d_G(u, v) = \infty$.

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The **eccentricity** of u in G, denoted $\mathrm{ecc}(u)$ or $\epsilon(u)$ is the length of the longest path with u as an endpoint, or

$$\mathrm{ecc}(u) = \max_{v \in V(G)} d(u, v).$$

The **diameter** of G, $\operatorname{diam}(G)$, is defined as

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The **center** of a graph G is the induced subgraph of G whose vertex set is the set of all vertices of eccentricity $\mathrm{rad}(G)$.

The center could be the whole graph.

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Proof. **Base:** $n \leq 2$.
Induction step. Suppose the theorem holds for all trees with less than n vertices. Take any tree T with n vertices.

Let L be the set of leaves in T and $T' = T - L$.

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Let L be the set of leaves in T and $T' = T - L$.

By Lemma 2.1 (b), T' is a tree. We claim that

$$\text{for each } u \in V(T'),\ \epsilon_{T'}(u) = \epsilon_T(u) - 1. \tag{1}$$

Indeed, each longest path in T starting from u ends at a leaf (which is not in T'). This shows inequality $\leq$.
On the other hand, if $uv_1v_2 \ldots v_{k-1}v_k$ is a longest path in T starting from u, then all vertices $u, v_1, v_2 \ldots, v_{k-1}$ are in T'. Hence we also have $\geq$.

By (1), the center of T' is the same as in T.

This proves Theorem 2.3.