JOURNAL OF COMBINATORIAL THEORY, Series B 63, 153–158 (1995)

The List Chromatic Index of a Bipartite Multigraph

FRED GALVIN*

Department of Mathematics, University of Kansas, Lawrence, Kansas 66045-2142

Received June 28, 1994

For a bipartite multigraph, the list chromatic index is equal to the chromatic index (which is, of course, the same as the maximum degree). This generalizes Janssen's result on complete bipartite graphs $K_{m,n}$ with $m \neq n$; in the case of $K_{n,n}$ it answers a question of Dinitz. (The list chromatic index of a multigraph is the least number n for which the edges can be colored so that adjacent edges get different colors, the color of each edge being chosen from an arbitrarily prescribed list of n different colors associated with that edge.) © 1995 Academic Press, Inc.

1. INTRODUCTION

The study of list coloring problems (i.e., graph coloring problems where each of the elements to be colored has its own list of permissible colors) was initiated by Vizing [9] and, independently but later, by Erdős, Rubin, and Taylor [3], whose terminology we follow.

Consider a graph G, with vertex set $V = V(G)$ and edge set $E = E(G)$, and two functions $f, g: V \to \mathbb{N}$, where $\mathbb{N}$ is the set of nonnegative integers. The graph G is *$(f:g)$-choosable* if, given any sets A_v $(v \in V)$ of "colors" with $|A_v| = f(v)$, we can choose subsets $B_v \subseteq A_v$ with $|B_v| = g(v)$ so that $B_u \cap B_v = \emptyset$ whenever $\{u, v\} \in E$. The graph is *n-choosable* if it is $(f:g)$-choosable for the constant functions $f(v) = n$, $g(v) = 1$. The *choice number* $\mathrm{ch}(G)$ is the least number n for which G is n-choosable. The *chromatic index* of a multigraph H is the chromatic number of the line graph $L(H)$; analogously, the *list chromatic index* of H is the choice number of $L(H)$. (The list chromatic index has also been called, rather confusingly, the "list chromatic number.")

Clearly, $\mathrm{ch}(G) \geqslant \chi(G)$ for every graph G; the example $\mathrm{ch}(K_{3,3}) = 3$ shows that the inequality can be strict [3, pp. 127, 145, 153]. It has been conjectured that $\mathrm{ch}(G) = \chi(G)$ whenever G is a line graph; in other words, that

* This project was sponsored by the National Security Agency under Grant Number MDA904-92-H-3037. The U.S. Government's right to retain a nonexclusive royalty-free license in and to the copyright covering this paper, for governmental purposes, is acknowledged.

0095-8956/95 $6.00
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the list chromatic index of a multigraph is always equal to the (ordinary) chromatic index. This is the *list coloring conjecture* (LCC); see Alon [1, p. 3] or Häggkvist and Chetwynd [6, p. 509] for the history. (Some authors state the LCC only for simple graphs; however, the formulation in [6] explicitly includes multigraphs.)

The LCC has been proved only in a few special cases. Janssen [7] proved it for the graphs $K_{m,n}$ with $m \neq n$. We generalize Janssen's result by proving the LCC for all bipartite multigraphs (Theorem 4.1). In particular, $L(K_{n,n})$ is n-choosable; this answers a question of Dinitz [3, p. 157]. The proof is very simple and uses no new ideas.

In Section 2 we relate an observation of Bondy, Boppana, and Siegel [2, p. 129] on the choosability of digraphs in which every induced subdigraph has a kernel; in particular, the choice number of such a digraph is at most $d+1$, where d is the maximum outdegree. In Section 3 we note that, if H is a bipartite multigraph, every "normal orientation" of $L(H)$ has a kernel; this is just the "stable marriage theorem" of Gale and Shapley [4, 5] as reformulated and generalized by Maffray [8]. In Section 4 we prove Theorem 4.1 by showing that, if H is bipartite with chromatic index n, then $L(H)$ has a normal orientation with maximum outdegree $n-1$; for this we use a generalization of the "Latin rectangle orientation" which was used implicitly by Alon and Tarsi [2, p. 132] and explicitly by Janssen [7, p. 245]. The stable marriage construction is incorporated in the proof of Theorem 4.1, which can be read independently of Section 3. In Section 5 we discuss the unsolved problem of finding the best possible choosability results for $L(K_{n,n})$; some unpublished results of Taylor are quoted by permission.

2. Kernels and Choosability

Consider a digraph $D = (V, \vec{E})$. We use the notation $u \to v$ to mean that $(u, v) \in \vec{E}$. We will assume that D is loopless and that any two vertices are joined by at most one arc in each direction; thus, the *outdegree* of a vertex v is $\text{od}(v) = |\{u : v \to u\}|$, and the *closed neighborhood* $\text{N}[v] = \{u : v \to u \text{ or } v = u\}$ has cardinality $|\text{N}[v]| = \text{od}(v) + 1$. The *underlying graph* of D is the graph $G = (V, E)$, where $E = \{\{u, v\} : u \to v \text{ or } v \to u\}$. The digraph D is *$(f:g)$-choosable* if its underlying graph is $(f:g)$-choosable. A *kernel* of D is an independent set $K \subseteq V$ such that, for each vertex $v \in V \backslash K$, there is a vertex $u \in K$ such that $v \to u$. A kernel of the subdigraph of D induced by a set $S \subseteq V$ will also be called a kernel of S.

Alon and Tarsi attributed the following observation to Bondy, Boppana, and Siegel [2, Remark 2.4, p. 129]; it was stated for the special case where D has no odd directed cycles and $g(v) = 1$, but the generalization is obvious.

LEMMA 2.1. (Bondy, Boppana, and Siegel). *Let D be a digraph in which every induced subdigraph has a kernel. If $f, g\colon V(D) \to \mathbb{N}$ are such that $f(v) \geq \sum_{u \in N[v]} g(u)$ whenever $g(v) > 0$, then D is $(f : g)$-choosable.*

Proof. Let $V = V(D)$. We use induction on $\sum_{v \in V} g(v)$. Let $W = \{v \in V: g(v) > 0\}$; we can assume that $W \neq \emptyset$. Let sets A_v ($v \in V$) with $|A_v| = f(v)$ be given. Choose $c \in \bigcup_{v \in W} A_v$, let $S = \{v \in W : c \in A_v\}$, and let K be a kernel of S. Define $g'\colon V \to \mathbb{N}$ by setting $g'(v) = g(v) - 1$ for $v \in K$ and $g'(v) = g(v)$ otherwise, and let $f'(v) = |A_v \setminus \{c\}|$. Then $\sum_{v \in V} g'(v) < \sum_{v \in V} g(v)$, and $f'(v) \geq \sum u \in N[v]\, g'(u)$ whenever $g'(v) > 0$. By the induction hypothesis, D is $(f' : g')$-choosable. Thus there are sets $B'_v \subseteq A_v \setminus \{c\}$, with $|B'_v| = g'(v)$, such that $B'_u \cap B'_v = \emptyset$ whenever u and v are adjacent. Define sets $B_v \subseteq A_v$ by setting $B_v = B'_v \cup \{c\}$ if $v \in K$ and $B_v = B'_v$ otherwise; then $|B_v| = g(v)$ for all $v \in V$, and $B_u \cap B_v = \emptyset$ if u and v are adjacent. ∎

A graph is *$(m : n)$-choosable* if it is $(f : g)$-choosable for the constant functions $f(v) = m$, $g(v) = n$. See Erdős, Rubin, and Taylor [3, p. 155] and Alon [1, pp. 22–23, 28–29] for more on $(m : n)$-choosability.

COROLLARY 2.2. *Let D be a digraph, with maximum outdegree $n - 1$, in which every induced subdigraph has a kernel. Then D is $(kn : k)$-choosable for every k; in particular, D is n-choosable.*

3. EXISTENCE OF KERNELS

A *clique* in a digraph is a nonempty set of vertices such that any two are joined by an arc in at least one direction. A digraph is *normal* if every clique has a kernel, which necessarily consists of a single vertex. An *orientation* of a graph G is any digraph (possibly containing antiparallel arcs) having G as its underlying graph. A graph is *solvable* if every normal orientation has a kernel. It is easy to see that every induced subgraph of a solvable graph is solvable. Maffray gave the following characterization of solvable line graphs [8, Theorem 1, p. 2].

THEOREM 3.1 (Maffray). *A line graph (of a multigraph) is solvable if and only if it is perfect.*

We will use only the following special case of Maffray's theorem.

COROLLARY 3.2. *The line graph of a bipartite multigraph is solvable. (Thus, if D is a normal orientation of the line graph of a bipartite multigraph, then every induced subdigraph of D has a kernel.)*

The "stable marriage theorem" of Gale and Shapley [4, 5] says, in our terminology, that the graph $K_{n,n}$ (hence every simple bipartite graph) has a solvable line graph. From this point of view, Corollary 3.2 is just the stable marriage theorem extended to multigraphs; moreover, the proof of this special case of Maffray's theorem is similar to the Gale–Shapley argument. Still, it was Maffray's formulation that motivated the present work; I am indebted to Professor Bondy for pointing out the connection with stable marriages.

4. The Bipartite LCC Proved

If K is a set of vertices in a digraph, we say that K *absorbs* a vertex v if $N[v] \cap K \neq \varnothing$, and that K absorbs a set S of vertices if K absorbs each vertex in S. Thus, a kernel of S is just an independent subset of S that absorbs S.

Theorem 4.1. *Let H be a bipartite multigraph, let $G = L(H)$, and suppose that G is n-colorable. Then G is $(kn : k)$-choosable for every k; in particular, G is n-choosable.*

Proof. Let $V = V(G) = E(H)$. Let (X, Y) be a bipartition of H. For $x \in V(H)$, we call the set $\{v \in V : v \text{ is incident with } x\}$ a *row* if $x \in X$, a *column* if $x \in Y$. Thus, two elements of V are adjacent just in case they are in the same row or the same column (or both). For $v \in V$, let $\mathrm{R}(v)$ and $\mathrm{C}(v)$ be the row and the column, respectively, containing v.

Let $f: V \to \{1, ..., n\}$ be a legal coloring, i.e., f is one-to-one on each row and column. Let D be the orientation of G in which $u \to v$ if either $\mathrm{R}(u) = \mathrm{R}(v)$ and $f(u) > f(v)$ or else $\mathrm{C}(u) = \mathrm{C}(v)$ and $f(u) < f(v)$. (If u and v are parallel edges in H, we have both $u \to v$ and $v \to u$.) It is easy to see that $\mathrm{od}(v) < n$ for each $v \in V$, since f is one-to-one on $\mathrm{N}[v]$. By Corollary 2.2, then, all that remains is to show that every induced subdigraph of D has a kernel. This follows immediately from Corollary 3.2, as D is clearly "normal"; for the convenience of the reader, here is a direct proof.

We show, by induction on $|S|$, that every set $S \subseteq V$ has a kernel. Given $S \subseteq V$, let $T = \{v \in S : f(v) < f(u) \text{ whenever } v \neq u \in \mathrm{R}(v) \cap S\}$. If T is independent, then T is a kernel of S, as T clearly absorbs S; thus we can assume that T is not independent. Then T has two elements in the same column: say $v_1, v_2 \in T$, $\mathrm{C}(v_1) = \mathrm{C}(v_2) = C$, $f(v_1) < f(v_2)$. Choose $v_0 \in C \cap S$ so that $f(v_0) < f(u)$ whenever $v_0 \neq u \in C \cap S$. By the definition of T and the choice of v_0, we have $\mathrm{N}[v_2] \cap S \subseteq C \cap S \subseteq \mathrm{N}[v_0]$. By the induction hypothesis, $S \setminus \{v_0\}$ has a kernel K. Since $v_2 \in S \setminus \{v_0\}$, it follows that K absorbs v_2, i.e., $\mathrm{N}[v_2] \cap K \neq \varnothing$. Since $\mathrm{N}[v_0] \cap K \supseteq \mathrm{N}[v_2] \cap K$, it follows that K also absorbs v_0, and so K is a kernel of S. ∎

Note, by the way, that the general statement of Theorem 4.1 can be derived from the case $k = 1$; just split each edge of H into k parallel edges.

COROLLARY 4.2. *The graph $L(K_{n,n})$ is $(kn : k)$-choosable for every k; in particular, it is n-choosable.*

The question, whether $L(K_{n,n})$ is n-choosable, was raised by J. Dinitz [3, p. 157]. According to Alon [1, p. 27], the answer was known for $n \leqslant 4$ and $n = 6$, having been proved by H. Taylor (unpublished) for $n = 3$ and by Alon and Tarsi [2, p. 132] for $n = 4$ and $n = 6$. Janssen, by proving the LCC for $K_{n,n+1}$, showed that $L(K_{n,n})$ is always $(n+1)$-choosable [7, Theorem 2.4, p. 248].

5. The Generalized Dinitz Problem

Let $G_n = L(K_{n,n})$. While Corollary 4.2 is "best possible" in a certain sense (G_n is not $(kn-1 : k)$-choosable for $k \geqslant 1$), many questions are left open. A graph is *f-choosable* if it is $(f : g)$-choosable for the constant function $g(v) = 1$. The following natural generalization of the Dinitz problem was proposed independently by (at least) H. Taylor and D. Knuth (personal communications) and the author.

Problem. For what functions f: $V(G_n) \to \mathbb{N}$ is G_n f-choosable?

By Lemma 2.1 and Corollary 3.2, G_n is f-choosable for the function $f(v) = \text{od}_D(v) + 1$, where D is any normal orientation of G_n. For $n = 2$, the functions obtained in this way are the minimal functions f for which G_n is f-choosable; however, for $n = 3$ the constant function $f(v) = 3$ is not minimal.

Consider a function f: $V(G_3) \to \{1, 2, 3\}$, and let $W_i = \{v : f(v) = i\}$. In unpublished work, Taylor proved that G_3 is f-choosable if $W_1 = \varnothing$ and $|W_2| = 1$; on the other hand, he found examples showing that G_3 is not f-choosable if $W_1 \neq \varnothing$, or if W_2 contains either two adjacent of three independent vertices. The remaining case has been settled by a tedious case analysis showing that G_3 is f-choosable if $W_1 = \varnothing$ and W_2 consists of two independent vertices.

Let k_n be the least k such that G_n is f-choosable when $f(v_0) = k$ for some vertex v_0 while $f(v) = n$ for $v \neq v_0$. It is known that $n/2 < k_n \leqslant n$ (lower bound due to Taylor) and $k_3 = 2$.

Problem. It is $k_n < n$ for all $n > 2$?

ACKNOWLEDGMENTS

I thank J. A. Bondy and F. Maffray for their helpful comments, H. Taylor for permission to quote his unpublished results, and G. Kangas for a stimulating presentation of the Janssen and Alon–Tarsi papers.

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