MATH 582, FALL 2021 – PROBLEM SET 1

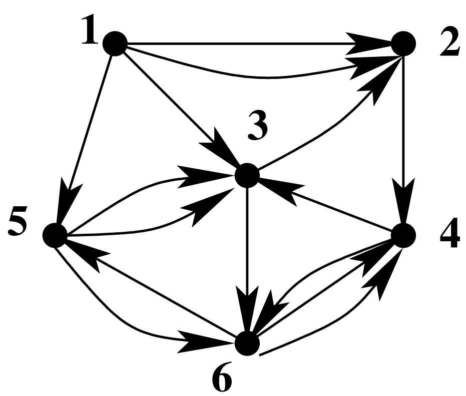
Do five of the six problems below. Due Wednesday, September 15.

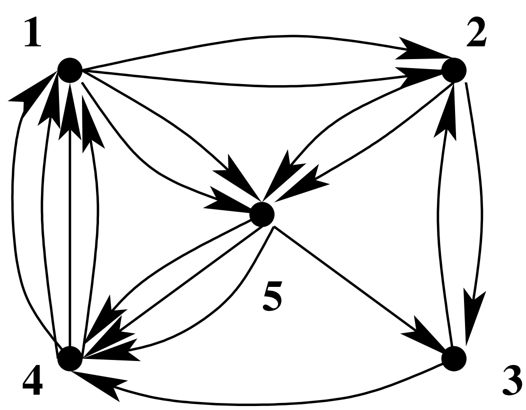
1. Let G and H be two tournaments on a vertex set V. Prove that $d^+_G(v) = d^+_H(v)$ for all $v \in V$ if and only if G can be turned into H by a sequence of direction-reversals on cycles of length 3. (Hint: Consider the graph formed by the edges with the opposite orientations in G and H.)

2. Use the Matrix Tree Theorem to find the number of spanning trees in the complete bipartite graph $K_{n,n+1}$.

3. Using the Directed Matrix Tree Theorem, find the number of spanning in-trees and out-trees with the root **6** in the picture below on the left.

4. Using the Matrix Arborescence Theorem, find in the same (left) picture the number of spanning out-trees with the root **1** in which **1** has out-degree 2. Find in the right picture the number of spanning out-trees with the root **2** in which **5** has out-degree 2.





5. Using the BEST Theorem, find the number of Eulerian circuits in the graph in the picture above on the right.

6. **Binet-Cauchy Formula.** Let $C = AB$, where A is an $n \times m$ matrix and B is an $m \times n$ matrix. Given $S \subseteq [m]$ with $|S| = n$, let A_S denote the $n \times n$ submatrix of A whose columns are indexed by S and let B_S denote the $n \times n$ submatrix of B whose rows are indexed by S. Prove that $\det C = \sum_S \det A_S \det B_S$, where the summation extends over all n-element subsets of $[m]$. (This completes the proof of the Matrix Tree Theorem.) (Hint: Consider the matrix equation $\begin{pmatrix} I_m & 0 \\ A & I_n \end{pmatrix}\begin{pmatrix} -I_m & B \\ A & 0 \end{pmatrix} = \begin{pmatrix} -I_m & B \\ 0 & AB \end{pmatrix}$.)