MATH 582, FALL 2021 – PROBLEM SET 2

Do five of the six problems below. Due Wednesday, October 6.

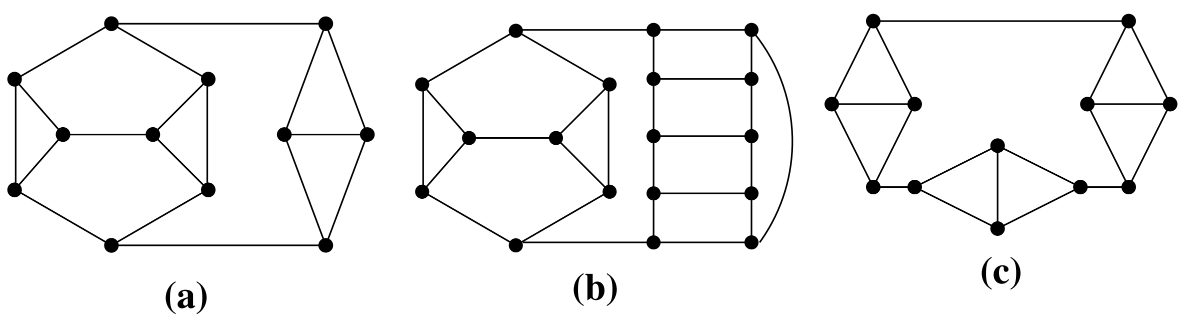
1. Prove the following statements about graceful graphs:
(a) A graceful graph may have a non-graceful component.
(b) A graph with all components graceful need not be graceful.
(c) K_4 is graceful.

2. Prove that if the Graceful Tree Conjecture is true and T is a tree with m edges, then K_{2m} can be decomposed into $2m-1$ copies of T. (Hint: Use proof of Theorem 6.1.41 for a suitable tree with $m-1$ edges.)

3. Let H be the Cartesian product of two 3-regular bipartite graphs G_1 and G_2. Prove that H decomposes into paths with six edges.

4. A partial case of the Corradi-Hajnal Theorem (which is also a partial case of Hajnal-Szemerédi Theorem which in turn is a partial case of the Bollobás-Eldridge-Catlin Conjecture) says that for every positive integer s, each $3s$-vertex graph G with minimum degree at least $2s$ contains a spanning subgraph all whose components are triangles.
(a) Using this fact, prove that for every positive integer s, each $(3s-1)$-vertex graph G with minimum degree at least $2s-1$ contains a spanning subgraph with one component K_2 and all other components being triangles.
(b) Using the above, prove that for every integer $s \geq 4$, each $(3s-1)$-vertex graph G with minimum degree at least $2s-1$ contains a spanning subgraph with two components $K_4 - e$ and all other components being triangles.

5. For each of the three graphs below, determine whether it has a nearly equitable 3-coloring or not. If it has, show it. If it hasn't, prove it.



6. (Bollobás' result) (a) Prove that each graph G with $\delta(G) \geq k-1$ contains each tree with k vertices.

(b) Let $n > k$ and $m > (k-1)(n-k/2)$. Prove that every n-vertex graph with at least m edges contains a subgraph with minimum degree at least k.

(c) Let $s := \lfloor n/\sqrt{2} \rfloor$. For $k = 2, 3, \ldots, s$, let T_k be a tree with k vertices. Using (b) and (a) show that $T_2, T_3, \ldots, T_s$ pack into K_n. (Hint: Pack larger trees first.)