1. Prove the following statements about graceful graphs:
   (a) A graceful graph may have a non-graceful component.
   (b) A graph with all components graceful need not be graceful.
   (c) $K_4$ is graceful.

2. Prove that if the Graceful Tree Conjecture is true and $T$ is a tree with $m$ edges, then $K_{2m}$ can be decomposed into $2m - 1$ copies of $T$. (Hint: Use proof of Theorem 6.1.41 for a suitable tree with $m - 1$ edges.)

3. Let $H$ be the Cartesian product of two 3-regular bipartite graphs $G_1$ and $G_2$. Prove that $H$ decomposes into paths with six edges.

4. A partial case of the Corradi-Hajnal Theorem (which is also a partial case of Hajnal-Szemerédi Theorem which in turn is a partial case of the Bollobás-Eldridge-Catlin Conjecture) says that for every positive integer $s$, each 3s-vertex graph $G$ with minimum degree at least $2s$ contains a spanning subgraph all whose components are triangles.
   (a) Using this fact, prove that for every positive integer $s$, each $(3s-1)$-vertex graph $G$ with minimum degree at least $2s-1$ contains a spanning subgraph with one component $K_2$ and all other components being triangles.
   (b) Using the above, prove that for every integer $s \geq 4$, each $(3s-1)$-vertex graph $G$ with minimum degree at least $2s-1$ contains a spanning subgraph with two components $K_4 - e$ and all other components being triangles.

5. For each of the three graphs below, determine whether it has a nearly equitable 3-coloring or not. If it has, show it. If it hasn’t, prove it.

6. (Bollobás’ result) (a) Prove that each graph $G$ with $\delta(G) \geq k - 1$ contains each tree with $k$ vertices.
   (b) Let $n > k$ and $m > (k-1)(n-k/2)$. Prove that every $n$-vertex graph with at least $m$ edges contains a subgraph with minimum degree at least $k$.
   (c) Let $s := \lceil n/\sqrt{2} \rceil$. For $k = 2, 3, \ldots, s$, let $T_k$ be a tree with $k$ vertices. Using (b) and (a) show that $T_2, T_3, \ldots, T_s$ pack into $K_n$. (Hint: Pack larger trees first.)