MATH 582, FALL 2021 – PROBLEM SET 4

Do five of the six problems below. Due Wednesday, November 17.

1. (a) Prove that if a graph $G$ is not $k$-linked, then the graph $G'$ obtained from $G$ by adding two all-adjacent vertices is not $(k + 1)$-linked. (b) Prove that every graph with minimum degree at least 24 contains a 2-linked subgraph.

2. A graph $G$ is $H$-linked if for every injection $f : V(H) \rightarrow V(G)$, there is an $H$-subdivision in $G$ that puts the vertex representing $v \in V(H)$ at $f(v)$ in $G$ for all $v \in V(H)$. Let $H$ be the graph consisting of two disjoint $K_2$ and three isolated vertices. What is the minimum connectivity that may have an $H$-linked graph? What is the maximum connectivity that may have a non-$H$-linked graph? (Hint: You may use all results stated (even without proofs) in Section 7.1 of the book.)

3. An expansion subdivides two edges and adds an edge joining the two new vertices. An erasure deletes an edge connecting two vertices of degree 3 and replaces each of the two obtained paths of length 2 by a single edge. (See the figure below.) Erasure is not allowed if it would produce multiple edges.

(a) Use 2-switches (recall Definition 6.2.6) and Theorem 6.2.7 to prove that every 3-regular graph can be obtained from $K_4$ by a sequence of expansions and erasures.

(b) For every $n$ divisible by 8, construct a 3-regular 2-connected $n$-vertex graph that cannot be obtained from a smaller 3-regular graph by expansion.

4. Prove that applying the expansion operation of Problem 3 to a 3-connected graph yields a 3-connected graph. Obtain the Petersen graph from $K_4$ by expansions. (Comment: Tutte proved that a 3-regular graph is 3-connected if and only if it arises from $K_4$ by a sequence of expansions.)

5. Given a graph $G$, a generalized vertex $k$-split forms a graph $H$ by replacing one vertex $x$ with a clique $X = \{x_1, \ldots, x_r\}$ such that $\bigcup_{i=1}^{r} N_H(x_i) = N_G(x) \cup \{x_1, \ldots, x_r\}$ and $d_H(x_i) \geq k$ for all $1 \leq i \leq r$.

(a) Prove that if $G$ is $k$-connected and

(*) for every $T \subseteq X$, the number of neighbors of $T$ in $N_G(x)$ is at least $k - r + |T|$, then $H$ is $k$-connected.

(b) Conclude that replacing a vertex $x$ in a 3-connected 3-regular graph with a
triangle and a matching joining the new vertices with the neighbors of \( x \) yields a 3-connected 3-regular graph.

6. Use ear decomposition to prove that every minimally 2-connected graph contains a vertex of degree 2. For \( n \geq 4 \), use this to prove that a minimally 2-connected \( n \)-vertex graph has at most \( 2n - 4 \) edges, with equality only for \( K_{2,n-2} \).