MATH 582, FALL 2021 – PROBLEM SET 5

Do five of the six problems below. Due Friday, December 3.

1. Use the Győri-Lovász Theorem to prove that for every positive integers k and n with $n \geq 3k$, every k-connected n-vertex graph
(a) has at least $k(n-2k+1)$ distinct matchings of size k;
(b) contains k vertex-disjoint paths of length two.

2. Complete the proofs of Lemmas 8.1.65, 8.1.66, and Theorem 8.1.67 in the book.

3. Let T_1, T_2, and T_3 be the three directed trees formed by the internal edges in a Schnyder labeling of a triangulation G. Let D be the digraph obtained by deleting from G the external edges and reversing the edges of T_1. Prove that D has no directed cycles.

4. a) Prove that every tree has a $(1, 2/3)$-separation.
b) Prove that every outerplanar graph has a $(2, 2/3)$-separation.
c) Prove that each grid with n vertices has a $\sqrt{n}$-separator with $\alpha = 1/2$.

5. For each of the 3 subgraphs of the Petersen Graph P below, determine whether it is planar or not:
(a) G_0, obtained from P by deleting two edges sharing a vertex;
(b) G_1, obtained from P by deleting two edges at distance 1;
(c) G_2, obtained from P by deleting two edges at distance 2.
"Determine" means you need to prove your answers. Just answers do not count.

6. A *normal plane map* is a connected plane multigraph in which all vertex and face degrees are at least 3. Prove that every normal plane map has an edge with the sum of the degrees of the ends at most 13. Give an example that this is sharp. (Hint: Reduce the problem to triangulations and use discharging but beware that you can add an edge xy only if $d(x) + d(y) \geq 12$.)