A model of reconstructing a whole object from its parts is *Graph reconstruction*. For a graph $G$, a *card* or vds is a subgraph $G - v$ for some $v \in V(G)$. The *deck* is the set of all cards of a graph. A graph is *reconstructible* if no other graph has the same deck.

**Examples:** A graph with 2 vertices and a graph with 5 vertices.

**Reconstruction Conjecture:** Every graph with at least 3 vertices is reconstructible. Proved for narrow classes.

A graph parameter is *reconstructible* if it can be computed from the deck when $n > 2$.

A class $\mathcal{G}$ of graphs is *recognizable* if the property of membership in $\mathcal{G}$ is reconstructible.

**Examples.** A *copy of $Q$* in $G$ is a subgraph of $G$ isomorphic to $Q$.

**Proof.** $s_Q(G) = \sum_{v \in V(G)} \frac{s_Q(G - v)}{n - |Q|}$, $s_Q(G, v) = s_Q(G) - s_Q(G - v)$. □

**Corollary:** Regular graphs are reconstructible.

**Theorem 2.2** (Kelly, 1957, Th. 6.3.13 in the book). *Disconnected graphs with at least 3 vertices are reconstructible.*

**Proof.** First we show that the class of disconnected graphs is recognizable. For this, observe that a graph $G$ is connected iff at least two of its vds are connected.

Now, if some card of a disconnected graph is connected, then this vertex is isolated and we see the rest of the graph in the card. If none of the cards is connected, choose a largest component over all cards, say $M$. Fix any subgraph $L$ of $M$ with $|V(L)| = |M| - 1$. Among the cards with the fewest copies of $M$, choose one with the most copies of $L$-components. Then we know all. □

A more complicated theorem is about reconstruction of trees. We need some notions and claims.

Recall that each tree has one or two adjacent centers. The *branches* of a bicentral tree are the component obtained by deleting the central edges. The branches of an unicentral tree $T$ with center $c$ are the components of $T - c$ with the added $c$ adjacent to its neighbor in $T$ in this component. They are *rooted trees* with the root in the center.

**Examples.**
When \( \Delta(G) > 2 \), \( \alpha(v) \) denotes the distance from \( v \) to the closest vertex of degree at least 3. A peripheral vertex is a vertex with largest eccentricity. An arm in a tree is a branch containing a peripheral vertex.

**Lemma 2.3.** Let \( n \geq 3 \).

(a) Trees, paths and trees of diameter \( d \) are recognizable.

(b) For a tree \( T \), the set \( \{ \alpha(v) \}_{v \in V(T)} \) is reconstructible.

**Proof.** Each tree is a connected graph with \( n - 1 \) edges. A path is a tree with max degree 2. If a tree is not a path, then we see the longest path in a card. This proves (a).

For (b), if \( T \) is a path, then \( \alpha(v) \) is not defined for all \( v \). Suppose not. For every vertex of degree at least 3, we know this, and this means \( \alpha(v) = 0 \). Suppose \( d(v) = 2 \).

Let \( Y_k \) be tree with \( k + 3 \) vertices obtained from the path with \( k + 2 \) vertices by duplicating one leaf. For each \( k < n - 3 \) and each \( v \) we know \( s_{Y_k}(T, v) \). The least \( k \) such that \( s_{Y_k}(T, v) > 0 \) (if exists) is \( \alpha(v) \). If such \( k \) does not exist, then since \( T \) is not a path, \( \alpha(v) = n - 3 \). \( \Box \)

**Theorem 2.4** (Kelly, 1957, Th. 6.3.19 in the book.). Trees with at least 3 vertices are reconstructible.

**Proof.** Let a deck \( D \) be given. By Lemma 2.3(a), we may assume that \( G \) is a tree distinct from the path. And we know its diameter. Since peripheral vertices are those that belong to a path of length \( diam(G) \) and have degree 1, we know the cards of peripheral vertices. Let \( \mathcal{P} \) be this set of cards.

Call a tree special if it has exactly two branches, and one is a path. If \( G - v \in \mathcal{P} \), then the arm containing \( v \) is a path iff \( \alpha(v) \geq \frac{diam(G)}{2} \). If in addition \( G \) is special, then \( \alpha(v) > \frac{diam(G)}{2} \).

Thus

\[
G \text{ is special } \iff \mathcal{P} \text{ has } G - v \text{ with } \alpha(v) > \frac{diam(G)}{2}.
\]

So we can recognize whether \( G \) is special. If yes, then reconstruct \( G \) from \( G - v \in \mathcal{P} \) by appending \( v \) to any path arm of \( G - v \). So, suppose not.

Let \( \mathcal{Q} = \{ G - v : diam(G - v) = diam(G) \text{ and } d(v) = 1 \} \). We now show that

\[
(2) \quad \forall \text{ arm } A \text{ there is a leaf } w \notin A \text{ s.t. } G - w \in \mathcal{Q}.
\]

Indeed, if for each leaf \( w \notin A \), \( diam(G - w) < diam(G) \), then only one leaf is not in \( A \); thus \( G \) is special.

Let \( A \) be a largest arm. By (2) some \( G - w \in \mathcal{Q} \) contains \( A \). Preserving diameter preserves the center. So, \( A \) is an arm in \( G - w \). Thus from \( \mathcal{Q} \) we see all largest arms of \( G \).

**Case 1:** \( A \) is a path arm. Then each arm in cards in \( \mathcal{Q} \) is a path arm. Take a connected card with the fewest path arms and append \( v \) to a slightly shorter branch that is a path.

**Case 2:** \( A \) is not a path. Then there is a leaf \( u \in A \) s.t. \( G - u \in \mathcal{Q} \). Let \( L = A - u \). Then \( L \) is an arm in \( G - u \), so in a card \( C \in \mathcal{Q} \) with the fewest arms isomorphic \( A \) and most cards isomorphic \( L \) we replace one \( L \) with \( A \). \( \Box \)

**Theorem 2.5** (Tutte, 1976, Th. 6.3.21 in the book.). For \( n \geq 3 \) and a graph \( G \) with \( n \) vertices, the parameters below are reconstructible.

(A) \( s_Q \) if \( Q \) is a spanning disconnected subgraph with \( \delta(Q) \geq 1 \).
(B) For \( k \geq 2 \), the number of spanning connected subgraphs of \( G \) whose blocks are \( B_1, \ldots, B_k \).

(C) The number of 2-connected spanning subgraphs of \( G \) with \( m \) edges.

Note: we do not see these subgraphs in the cards.

Proof of (A). Suppose \( Q_1, \ldots, Q_k \) are the components of \( Q \).
For a graph \( H \), define \( b_Q(H) = \# \) of ways to express \( H \) as the union of \( Q_1, \ldots, Q_k \).

Example: \( Q_1 = K_3, Q_2 = P_3, Q_3 = K_2, H_1 = K_4 - e, H_2 = K_4 \). Then \( b_Q(H_1) = 2(5 + 4) = 18 \) and \( b_Q(H_2) = 12 \).

Important equality is:

\[
\prod_{i=1}^{k} s_{Q_i}(G) = \sum_{\delta(H) \geq 1} b_Q(H)s_H(G).
\]

Given any \( H \), we know \( b_Q(H) \). If \( |V(H)| \leq n - 1 \), then we know \( s_H(G) \). So, from (3) we know \( s_Q(G) \).

Proof of (B). Suppose \( B = \{ B_1, \ldots, B_k \} \) is the list of blocks, and \( n_i = |V(B_i)| \). Each connected graph with blocks \( B_1, \ldots, B_k \) has \( \sum_{i=1}^{k} n_i - k + 1 \) vertices.

For a graph \( H \), define \( b_B(H) = \# \) of ways to express \( H \) as the union of \( B_1, \ldots, B_k \). Again (3) with \( B \) in place of \( Q \) holds. We know: (a) \( b_B(H) \) for all \( H \), (b) \( s_H(G) \) when \( |V(H)| < n \) or \( H \) is disconnected.

Let \( S \) be the class of connected spanning subgraphs of \( G \) whose blocks are \( B_1, \ldots, B_k \). So, unknown are the values of \( s_H(G) \) when \( H \in S \). We do not find each of them, but want to find \( \sum_{H \in S} s_H(G) \). We know that for all such \( H, b_B(H) \) is the same: it is 1 when all \( B_i \) are distinct, and otherwise it is \( (m_1!) \ldots (m_j!) \) when they form \( j \) isomorphism classes.

Proof of (C). There are \( \binom{|E(G)|}{m} \) subgraphs of \( G \) with \( m \) edges. By Kelley’s Lemma we know the number of them with isolated vertices. By (A), we know the number of other disconnected subgraphs with \( m \) edges. By (B), we know the number of connected subgraphs with \( m \) edges and with cut vertices. \( \square \)

Corollary. The number of hamiltonian cycles and the number of spanning trees in a graph are reconstructible.

Bollobás result on 3 cards.

Edge-reconstruction, examples with 3 edges.

Edge-Reconstruction Conjecture (Harary, 1964): Every graph with more than 3 edges is edge-reconstructible. ——————————– Here Lecture 14 ended.

Lemma 2.6 (Edge-Kelly Lemma). Let \( m \geq 4 \). If \( |E(G)| = m > |E(Q)| \), then \( s_Q(G) \) is reconstructible.