2. Lecture notes: Reconstruction

A model of reconstructing a whole object from its parts is *Graph reconstruction*.

For a graph G, a *card* or *vds* is a subgraph $G - v$ for some $v \in V(G)$. The *deck* is the set of all cards of a graph. A graph is *reconstructible* if no other graph has the same deck.

Examples: A graph with 2 vertices and a graph with 5 vertices.

Reconstruction Conjecture: Every graph with at least 3 vertices is reconstructible.

Proved for narrow classes.

A graph parameter is *reconstructible* if it can be computed from the deck when $n > 2$.

A class $\mathcal{G}$ of graphs is *recognizable* if the property of membership in $\mathcal{G}$ is reconstructible.

Examples.

A *copy* of Q in G is a subgraph of G isomorphic to Q.

Examples.

Let $s_Q(G) =$ be # of copies of Q in G,
$s^*_Q(G) =$ be # of induced copies of Q in G,
$s_Q(G, v) =$ be # of copies of Q in G containing v,
$s^*_Q(G, v) =$ be # of induced copies of Q in G containing v.

Theorem 2.1 (Kelly's Lemma, Kelly, 1957, Lem. 6.3.6 in the book)**.** *If $n > 2$, $v \in V(G)$ and $|V(Q)| < |V(G)$, then all of $s_Q(G)$, $s^*_Q(G)$, $s_Q(G, v)$ and $s^*_Q(G, v)$ are reconstructible. In particular, the degrees sequence and the number of edges are reconstructible.*

Proof. $s_Q(G) = \frac{\sum_{v \in V(G)} s_Q(G-v)}{n-|Q|}$, $s_Q(G, v) = s_Q(G) - s_Q(G - v)$. □

Corollary: *Regular graphs are reconstructible.*

Theorem 2.2 (Kelly, 1957, Th. 6.3.13 in the book)**.** *Disconnected graphs with at least 3 vertices are reconstructible.*

Proof. First we show that the class of disconnected graphs is recognizable. For this, observe that a graph G is connected iff at least two of its vds are connected.

Now, if some card of a disconnected graph is connected, then this vertex is isolated and we see the rest of the graph in the card. If none of the cards is connected, choose a largest component over all cards, say M. Fix any subgraph L of M with $|V(L)| = |M| - 1$. Among the cards with the fewest copies of M, choose one with the most copies of L-components. Then we know all. □

A more complicated theorem is about reconstruction of trees. We need some notions and claims.

——————————— **Here Lecture 12 ended.**

Recall that each tree has one or two adjacent centers. The *branches* of a bicentral tree are the component obtained by deleting the central edges. The branches of an unicentral tree T with center c are the components of $T - c$ with the added c adjacent to its neighbor in T in this component. They are *rooted trees* with the root in the center.

Examples.

When $\Delta(G) > 2$, $\alpha(v)$ denotes the distance from v to the closest vertex of degree at least 3. A *peripheral* vertex is a vertex with largest eccentricity. An *arm* in a tree is a branch containing a peripheral vertex.

Lemma 2.3. *Let $n \geq 3$.*
(a) Trees, paths and trees of diameter d are recognizable.
(b) For a tree T, the set $\{\alpha(v)\}_{v \in V(T)}$ is reconstructible.

Proof. Each tree is a connected graph with $n-1$ edges. A path is a tree with max degree 2. If a tree is not a path, then we see the longest path in a card. This proves (a).

For (b), if T is a path, then $\alpha(v)$ is not defined for all v. Suppose not. For every vertex of degree at least 3, we know this, and this means $\alpha(v) = 0$. Suppose $d(v) = 2$.

Let Y_k be tree with $k+3$ vertices obtained from the path with $k+2$ vertices by duplicating one leaf. For each $k < n-3$ and each v we know $s_{Y_k}(T, v)$. The least k such that $s_{Y_k}(T, v) > 0$ (if exists) is $\alpha(v)$. If such k does not exist, then since T is not a path, $\alpha(v) = n - 3$. $\square$

Theorem 2.4 (Kelly, 1957, Th. 6.3.19 in the book.)**.** *Trees with at least* 3 *vertices are reconstructible.*

Proof. Let a deck $\mathcal{D}$ be given. By Lemma 2.3(a), we may assume that G is a tree distinct from the path. And we know its diameter. Since peripheral vertices are those that belong to a path of length $diam(G)$ and have degree 1, we know the cards of peripheral vertices. Let $\mathcal{P}$ be this set of cards.

Call a tree *special* if it has exactly two branches, and one is a path. If $G - v \in \mathcal{P}$, then the arm containing v is a path iff $\alpha(v) \geq \frac{diam(G)}{2}$. If in addition G is special, then $\alpha(v) > \frac{diam(G)}{2}$. Thus

$$(1) \qquad G \text{ is special} \Leftrightarrow \mathcal{P} \text{ has } G - v \text{ with } \alpha(v) > \frac{diam(G)}{2}.$$

So we can recognize whether G is special. If yes, then reconstruct G from $G - v \in \mathcal{P}$ by appending v to any path arm of $G - v$. So, suppose not.

Let $\mathcal{Q} = \{G - v : diam(G - v) = diam(G) \text{ and } d(v) = 1\}$. We now show that

$$(2) \qquad \forall \text{ arm } A \text{ there is a leaf } w \notin A \text{ s.t. } G - w \in \mathcal{Q}.$$

Indeed, if for each leaf $w \notin A$, $diam(G - w) < diam(G)$, then only one leaf is not in A; thus G is special.

Let A be a largest arm. By (2) some $G - w \in \mathcal{Q}$ contains A. Preserving diameter preserves the center. So, A is an arm in $G - w$. Thus from $\mathcal{Q}$ we see all largest arms of G.

Case 1: A is a path arm. Then each arm in cards in $\mathcal{Q}$ is a path arm. Take a connected card with the fewest path arms and append v to a slightly shorter branch that is a path.

Case 2: A is not a path. Then there is a leaf $u \in A$ s.t. $G - u \in \mathcal{Q}$. Let $L = A - u$. Then L is an arm in $G - u$, so in a card $C \in \mathcal{Q}$ with the fewest arms isomorphic A and most cards isomorphic L we replace one L with A. $\square$

——————————— **Here Lecture 13 ended.**

Theorem 2.5 (Tutte, 1976, Th. 6.3.21 in the book.)**.** *For $n \geq 3$ a graph G with n vertices, the parameters below are reconstructible.*

(A) s_Q if Q is a spanning disconnected subgraph with $\delta(Q)) \geq 1$.

(B) For $k \geq 2$, the number of spanning connected subgraphs of G whose blocks are $B_1, \ldots, B_k$.
(C) The number of 2-connected spanning subgraphs of G with m edges.

Note: we do not see these subgraphs in the cards.
Proof of (A). Suppose $Q_1, \ldots, Q_k$ are the components of Q.
For a graph H, define $b_Q(H) =\#$ of ways to express H as the union of $Q_1, \ldots, Q_k$.
Example: $Q_1 = K_3$, $Q_2 = P_3$, $Q_3 = K_2$, $H_1 = K_4 - e$, $H_2 = K_4$. Then $b_Q(H_1) = 2(5+4) = 18$ and $b_Q(H_2) = 12$.
Important equality is:

$$\prod_{i=1}^{k} s_{Q_i}(G) = \sum_{H \subseteq G, \delta(H) \geq 1} b_Q(H) s_H(G). \tag{3}$$

Given any H, we know $b_Q(H)$. If $|V(H)| \leq n-1$, then we know $s_H(G)$. So, from (3) we know $s_Q(G)$.
Proof of (B). Suppose $\mathbf{B} = \{B_1, \ldots, B_k\}$ is the list of blocks, and $n_i = |V(B_i)|$. Each connected graph with blocks $B_1, \ldots, B_k$ has $\sum_{i=1}^{k} n_i - k + 1$ vertices.
For a graph H, define $b_{\mathbf{B}}(H) =\#$ of ways to express H as the union of $B_1, \ldots, B_k$. Again (3) with $\mathbf{B}$ in place of Q holds. We know: (a) $b_{\mathbf{B}}(H)$ for all H, (b) $s_H(G)$ when $|V(H)| < n$ or H is disconnected.
Let S be the class of connected spanning subgraphs of G whose blocks are $B_1, \ldots, B_k$. So, unknown are the values of $s_H(G)$ when $H \in S$. We do not find each of them, but want to find $\sum_{H \in S} s_H(G)$. We know that for all such H, $b_{\mathbf{B}}(H)$ is the same: it is 1 when all B_i are distinct, and otherwise it is $(m_1!) \ldots (m_j!)$ when they form j isomorphism classes.
Proof of (C). There are $\binom{|E(G)|}{m}$ subgraphs of G with m edges. By Kelley's Lemma we know the number of them with isolated vertices. By (A), we know the number of other disconnected subgraphs with m edges. By (B), we know the number of connected subgraphs with m edges and with cut vertices. □

Corollary. *The number of hamiltonian cycles and the number of spanning trees in a graph are reconstructible.*

Bollobás result on 3 cards.

Edge-reconstruction, examples with 3 edges.
Edge-Reconstruction Conjecture (Harary, 1964): Every graph with more than 3 edges is edge-reconstructible.
——————————— **Here Lecture 14 ended.**

Lemma 2.6 (Edge-Kelly Lemma)**.** *Let $m \geq 4$. If $|E(G)| = m > |E(Q)|$, then $s_Q(G)$ is reconstructible.*

Proof. The same as for Kelly Lemma. □

Let $s'_Q(G)$ be # of injections $f : V(Q) \to V(G)$ s.t. edges of Q go to edges of G.
Then $s'_Q(G) = a(Q) \cdot s_Q(G)$, where $a(Q)$ is the number of automorphisms of Q.

Theorem 2.7 (Lovász, 1972), Th. 6.3.31 in the book)**.** *Let G be an n-vertex graph with m edges. If $m > \frac{1}{2}\binom{n}{2}$, then G is edge-reconstructible.*

Proof. We look at n-vertex graphs. For a graph Q, let $\mathcal{Q}(Q)$ be the set of all $2^{|E(Q)|}$ spanning subgraphs of Q. By inclusion-exclusion, for each G,

$$s'_Q(\overline{G}) = \sum_{F\in\mathcal{Q}(Q)} (-1)^{|E(F)|}\; s'_F(G). \tag{4}$$

Suppose n-vertex m-edge graph G_1 has the same edge deck as G. By (4),

$$s'_{G_1}(\overline{G}) = \sum_{F\in\mathcal{Q}(G_1)} (-1)^{|E(F)|}\; s'_F(G) \tag{5}$$

and

$$s'_G(\overline{G}) = \sum_{F\in\mathcal{Q}(G)} (-1)^{|E(F)|}\; s'_F(G). \tag{6}$$

The terms in RHSs of (5) and (6) containing F distinct from G_1 and G are the same. Also, both LHSs are zeros, since $|E(G)| > |E(\overline{G})|$. So $s'_{G_1}(G) = s'_G(G) > 0$, which means $G_1 = G_2$. $\square$

For a spanning subgraph R of Q, let $s'_{R:Q}(G)$ denote the number of injections $f : V(Q) \to V(G)$ s.t. the edges in R map into edges of G and the edges in $Q - E(R)$ map into non-edges of G.

Theorem 2.8 (Nash-Williams, 1976), Th. 6.3.33 in the book)**.** *If a graph G with at least 4 edges has a spanning subgraph R satisfying one of the properties below, then G is edge-reconstructible.*

1) $s'_{R:G}(H) = s'_{R:G}(G)$ for all H with the same edge deck as G.

2) $|E(G)| - |E(R)|$ is even and $s'_{R:G}(G) = 0$.

Corollary 2.9 (Müller, 1977), Cor. 6.3.34 in the book)**.** *Every graph G with $n \geq 4$ vertices and at least $1 + \log_2(n!)$ edges is edge-reconstructible.*

Proof of Corollary 2.9 modulo Th. 2.8. Let $m = |E(G)| \geq 1 + \log_2(n!)$. G has 2^{m-1} spanning subgraphs R s.t. $m - |E(R)|$ is even.

There are $n!$ injections $V(G) \to V(G)$; they preserve at most $n!$ sets R. If $2^{m-1} > n!$, then some R is never preserves, that is, $s'_{R:G}(G) = 0$. Apply part 2) of Th. 2.8. $\square$

——————————— **Here Lecture 15 ended.**

Proof of Th. 2.8. For a spanning subgraph R of G, let $\mathcal{Q} = \mathcal{Q}(R)$ be the set of spanning subgraphs of G containing R.

For every graph F, $s'_R(F) = \sum_{P\in\mathcal{Q}} s'_{P:G}(F)$. So, by Inclusion-Exclusion,

$$s'_{R:G}(F) = \sum_{P\in\mathcal{Q}} (-1)^{|E(P)|-|E(R)|}\; s'_P(F). \tag{7}$$

Let G' have the same edge deck as G. Consider $s'_{R:G}(G)$ and $s'_{R:G}(G')$. By edge-Kelly Lemma, almost all terms in RHS of (7) coincide, so

$$s'_{R:G}(G) - s'_{R:G}(G') = (-1)^{|E(G)|-|E(R)|}(s'_G(G) - s'_G(G')). \tag{8}$$

So if condition 1) of the theorem holds, then the LHS of (8) is 0.

If $s'_{R:G}(G) = 0$, then LHS ≤ 0. So if in addition $|E(G)| - |E(R)|$ is even, then $s'_G(G') \geq s'_G(G) > 0$. $\square$

3. Connectivity

3.1. **New min-max theorems.** Definitions and examples. Recollecting Menger Theorems, Expansion Lemma.

An r-*branching* in a digraph is an out-tree rooted at r.

Let $\kappa'(r,G)$ be the minimum # of edges whose deletion makes some $v \in V(G)$ unreachable from r.

For $X \subset V(G)$, let $F(X) =$ # of edges entering X. So

$$\kappa'(r,G) = \min\{F(X) : X \neq \emptyset, r \notin X\}.$$

Let $b(r,G) =$ max # of edge-disjoint r-branchings in G.

Theorem 3.1 (Edmonds, 1973, Th. 7.1.37 in the book)**.** *For each digraph G and each $r \in V(G)$, $b(r,G) = \kappa'(r,G)$.*

Proof. Let $k = \kappa'(r,G)$. The fact $b(r,G) \leq k$ is evident. We prove $b(r,G) \geq k$ by induction on k. The case $k = 1$ is clear.

———————————— **Here Lecture 16 ended.**

For the induction step, we will find an r-branching T s.t. $\kappa'(r, G - E(T)) \geq k - 1$.

Claim 1: *For all $U, W \subseteq V(G)$, $F(U) + F(W) \geq F(U \cup W) + F(U \cap W)$.* Proof in class.

A *partial r-branching* (p.b. for short) is an out-tree with root r. A p.b. is *good* if $\kappa'(r, G - E(B)) \geq k - 1$.

A p. b. with one edge is good. Let B be a largest good p.b. If for every $W \subset V(G)$ s.t. (a) $r \notin B$ and (b) $W \not\subseteq B$ we have $F_{G-E(B)}(W) \geq k$, then adding any edge from $V(B)$ to $V(G) - V(B)$ we get a good p.b. contradicting the choice of B.

So, choose a minimum $U \subset V(G)$ satisfying (a) and (b) s.t. $F_{G-E(B)}(U) = k - 1$. Since no edges entering $U - V(B)$ were deleted, $F_{G-E(B)}(U - V(B)) \geq k$.

Draw a picture !!

But $F_{G-E(B)}(U) = k - 1$. So there is $xy \in E(G)$ s.t. $x \in V(B) \cap U$ and $y \in U - V(B)$. Let $B' = B + xy$. By the maximality of B, $\kappa'(r, G - E(B')) \leq k - 2$. This means there is $W \subseteq V - r$ s.t.

$$F_{G-E(B')}(W) \leq k - 2.$$

This in turn means $F_{G-E(B)}(W) = k - 1$ and xy enters W, i.e. $x \notin W$ and $y \in W$. In particular, $U \cap W \neq U$. By Claim 1,

$$F_{G-E(B)}(W \cap U) + F_{G-E(B)}(W \cup U) \leq F_{G-E(B)}(W) + F_{G-E(B)}(U) = 2(k-1).$$

It follows that $F_{G-E(B)}(W \cap U) = F_{G-E(B)}(W \cup U) = k - 1$. This contradicts the choice of U. $\square$

Corollary 3.2 (Cor. 7.1.38 in the book)**.** *For each digraph G and any $r \in V(G)$, TFAE:*

(A) G has k pairwise edge-disjoint r-branchings.

(B) $\kappa'(r,G) \geq k$.

(C) For each $s \in V(G) - r$, $\exists\, k$ pairwise edge-disjoint r, s-paths.

(D) The underlying undirected H has k pairwise edge-disjoint spanning trees whose union G' is s.t. each vertex apart from r is entered by exactly k edges.

Proof. (A) $\Rightarrow$ (C) Evident. (C) $\Rightarrow$ (B) All r, s-paths should be broken.
(B) $\Rightarrow$ (A) Theorem 3.1. (A) $\Rightarrow$ (D) Evident.
(D) $\Rightarrow$ (B) Let $U \subseteq V - r$. Each spanning tree has at most $|U| - 1$ edges inside U, so $|E_{G'}(U)| \le k(|U| - 1)$. But althogher there are $k|U|$ edges entering the vertices in U. $\square$

Theorem 3.3 (Seymour, 1977, Th. 7.1.39 in the book)**.** *Theorem 3.1 implies the edge local directed version of Menger's Theorem.*

Proof. Let $x, y \in V(G)$ and $k = \kappa'(x, y)$. By the definition of k, for each $U \subseteq V(G) - x$ with $y \in U$, $F_G(U) \ge k$.

Let G' be obtained from G by adding k edges yz for each $z \in V(G) - x - y$. Then $F_{G'}(U) \ge k$ for each U not containing x. So by Theorem 3.1, G' has k edge-disjoint x-branchings. Each of them contains an x, y-path, and this path is contained in G. $\square$

——————————— **Here Lecture 17 ended.**

A *dicut* in a digraph G is an ordered partition $[S, \overline{S}]$ of $V(G)$, s.t. G has no edges from $\overline{S}$ to S.

By definition, a digraph is strongly connected iff it has no dicuts.

If the underlying undirected graph $\underline{G}$ is connected, then each dicut $[S, \overline{S}]$ has edge(s) from S to $\overline{S}$. Such edges we will call the *edges of* $[S, \overline{S}]$.

If we add to G a set L of directed edges s.t. for each dicut $[S, \overline{S}]$, L contains an edge from $\overline{S}$ to S, then $G + L$ is strongly connected. Certainly, for a digraph G with the underlying undirected graph $\underline{G}$ connected, the number of edges in such L must be at least the maximum number $m(G)$ of pairwise disjoint dicuts in G. We will prove a theorem by Lucchesi and Younger that one can find such L of size $m(G)$ with the property that each edge in L is a reversed edge from G. It was a conjecture by Younger and Robertson.

For the proof, we need a technical lemma by Lovász.

Lemma 3.4 (Lovász, 1976), Lem. 7.1.46 in the book)**.** *Let G be a digraph with at most k pairwise disjoint dicuts. If $D_1, \ldots, D_\ell$ are dicuts that together cover each edge of G at most twice, then $\ell \le 2k$.*

Note that the same dicut may appear twice among $D_1, \ldots, D_\ell$.

Theorem 3.5 (Lucchesi and Younger, 1978, Th. 7.1.47 in the book)**.** *For a digraph G with the underlying undirected graph $\underline{G}$ connected, the minimum number of edges in a set covering all dicuts equals the maximum number $m(G)$ of pairwise disjoint dicuts in G.*

Proof modulo Lemma 3.4. By induction on $m(G)$. If $m(G) = 0$, the claim is trivial. Suppose the theorem holds for all G' with $m(G') \le k - 1$. Let G be any digraph with $m(G) = k$.

Definitions of subdivisions and contractions: $G \oplus e$ and G/e.

Let $D = (S, \overline{S})$ be a dicut in a set of k pairwise disjoint dicuts in G. We subdivide edges of D one by one until subdividing any other edges from D would increase the number of pairwise disjoint dicuts. Suppose the resulting digraph is H, and e is an edge in D s.t. $H \oplus e$ has $k+1$ disjoint dicuts, say $D_1, \ldots, D_{k+1}$. We can consider those as dicuts in H such that D_1 and D_2 share e, but in each other pair of dicuts the dicuts are disjoint.

Consider $H' = H/e$. If H' has only $k-1$ disjoint dicuts, then G/e also has at most $k-1$ disjoint dicuts, hence by induction has a set S of $k-1$ edges covering all dicuts in G/e. But then S covers all dicuts in G that do not contain e. Thus $S+e$ covers all dicuts in G, a contradiction.

Hence H' has k disjoint dicuts, say $C_1, \ldots, C_k$. Those are disjoint dicuts in H not containing e. Then $\{C_1, \ldots, C_k, D_1, \ldots, D_{k+1}\}$ is a set of $2k+1$ dicuts in H contradicting Lemma 3.4. □

For the proof of Lemma 3.4, we will need some notation.

Sets A and B in a universe U are *crossing* if all of $A \cap B, A - B, B - A$ and $\overline{A \cup B}$ are non-empty. A family of sets is *laminar* if no two members are crossing.

Proof of Lemma 3.4. Let $k \geq 1$. Suppose a digraph G with at most k pairwise disjoint dicuts has dicuts $D_1, \ldots, D_{2k+1}$ that together cover each edge of G at most twice. Let $D_i = (S_i, \overline{S}_i)$ for $1 \leq i \leq 2k+1$.

Choose such a set with the maximum $\sum_{i=1}^{2k+1} |S_i|^2$. We claim that

(9) $$\textit{the family } \mathcal{S} = \{S_1, \ldots, S_{2k+1}\} \textit{ is laminar.}$$

Indeed, if S_1 and S_2 cross, replace D_1 and D_2 with pairs $D_1' = (S_1 \cap S_2, \overline{S_1 \cap S_2})$ and $D_2' = (S_1 \cup S_2, \overline{S_1 \cup S_2})$. We check (using pictures!) that (a) D_1' are D_2' are dicuts, (b) each edge of is covered at most twice by $D_1', D_2', D_3, \ldots, D_{2k+1}$, and
(c) $|S_1 \cap S_2|^2 + |S_1 \cup S_2|^2 > |S_1|^2 + |S_2|^2$. This proves (9).

——————————— **Here Lecture 18 ended.**

——————————— **Lecture 19 was by Prof. West on graph reconstruction.**

Consider the auxiliary graph H with $V(H) = U = \{D_1, \ldots, D_{2k+1}\}$, and $D_iD_j \in E(H)$ iff D_i and D_j share an edge. By the definition of k, $\alpha(H) \leq k$. We will prove that

(10) $$H \textit{ is bipartite.}$$

That would imply $|V(H)| \leq 2\alpha(H) \leq 2k$, a contradiction.

So suppose $C = D_1, \ldots, D_m, D_1$ is an odd cycle in H. If some D_i appears twice in C, then the edges of D_i do not belong to other D_js, a contradiction. So, all $D_1, \ldots, D_m$ are distinct, hence all $S_1, \ldots, S_m$ are distinct.

Since $D_i \cap D_{i+1} \neq \emptyset$, $S_i \cap S_{i+1} \neq \emptyset$ and $\overline{S}_i \cap \overline{S}_{i+1} \neq \emptyset$. Since $\{S_i, S_{i+1}\}$ is non-crossing,

(11) $$\textit{either } S_i \subset S_{i+1} \textit{ or } S_i \supset S_{i+1}.$$

Since m is odd, the condition cannot alternate all the time. So, we may assume

(12) $$S_m \subset S_1 \subset S_2.$$

Let j be the largest index s.t. S_1 *contains neither* S_j *nor* $\overline{S}_j$. By (12), j is well defined and $j \leq m-1$.

By (9), (*) either $S_1 \subset S_j$ or $S_1 \subset \overline{S_j}$. Let $e = xy \in D_j \cap D_{j+1}$.

Pictures!!

Rewriting (*), we have

(a) Either $S_1 \subset S_j$ or $S_1 \cap S_j = \emptyset$. Similarly,

(b) either $S_1 \supset S_{j+1}$ or $S_1 \cup S_{j+1} = U$, and

(c) both $S_j \cap S_{j+1} \neq \emptyset$ and $S_j \cup S_{j+1} \neq U$.

By (11), we have two cases. (and watch how e goes).

Case 1: $S_j \subset S_{j+1}$. By (a) we have two subcases.

Case 1.1: $S_1 \subset S_j$. (Picture!) Then $S_1 \not\supset S_{j+1}$, so by (b), $S_1 \supset \overline{S}_{j+1}$. But S_1 does not contain y.

Case 1.2: $S_1 \subset \overline{S}_j$. (Picture!) Again, $S_1 \not\supset S_{j+1}$, so by (b), $S_1 \supset \overline{S}_{j+1}$. But S_1 does not contain x.

Case 2: $S_j \supset S_{j+1}$. By (a) we have two subcases.

Case 2.1: $S_1 \subset S_j$. (Picture!) Then $S_1 \not\supset \overline{S}_{j+1}$, since $y \notin S_j \supset S_1$. So by (b), $S_1 \supset S_{j+1}$. But then $e \in D_j \cap D_{j+1} \cap D_1$.

Case 2.2: $S_1 \subset \overline{S}_j$. (Picture!) Then $S_1 \not\supset S_{j+1}$ and $S_1 \not\supset \overline{S}_{j+1}$, contradicting (b). □

3.2. **On k-linked graphs.** A graph G with at least $2k$ vertices is *k-linked*, if for any distinct $a_1, \ldots, a_k, b_1, \ldots, b_k \in V(G)$, there are k disjoint paths $P_1, \ldots, P_k$ s.t. $\forall i$, P_i is an a_i, b_i-path.

An example of a 5-connected but not 2-linked graph.

Jung: each non-planar 4-connected graph is 2-linked. So, each 6-connected graph is 2-linked.

For each fixed k, there is an $O(n^3)$-algorithm checking whether an n-vertex G is k-linked. For general k — NP-hard.

Before continuing of k-linked graphs, we digress on subdivisions. Recall the definition! Also, F-subdivisions.

——————————— **Here Lecture 20 ended.**

Theorem 3.6 (Mader, Thomassen, Th. 7.1.53 in the book)**.** *Let F have m edges and no isolated vertices. If a graph G has at least $|V(F)|$ vertices and $\delta(G) \geq 2^{m-1}$, then G contains an F-subdivision.*

We will use the lemma below:

Lemma 3.7 (Mader, Thomassen, Lem. 7.1.52 in the book)**.** *If $\delta(G) \geq 2k$, then G contains vertex disjoint subgraphs G' and H s.t. (1) $\delta(G') \geq k$, (2) each $v \in V(G')$ has a neighbor in H and (3) H is connected.*

Proof of Theorem 3.6 modulo Lemma 3.7. By induction on m. Check for $m = 1, 2$. Suppose $m \geq 3$ and the theorem is proved for $m - 1$.

If there is $xy \in E(F)$ with $d(x) = d(y) = 1$, then $F' = F - x - y$. In this case, choose any edge $uv \in E(G)$ and let $G' = G - u - v$.

Otherwise, let G' and H satisfy Lemma 3.7, and define F' as follows. If there is $xy \in E(F)$ with $d(x) \geq 2$ and $d(y) = 1$, then let $F' = F - y$. Otherwise $\delta(F) \geq 2$. Take any $xy \in E(F)$ and let $F' = F - xy$.

We claim that G' satisfies conditions for F'. Indeed, if $d(x) = d(y) = 1$, then $\delta(G') \geq \delta(G) - 2 \geq 2^{m-1} - 2 \geq 2^{m-2}$. Also in this case $|V(G')| = |V(G)| - 2 \geq |V(F)| - 2$.

In other cases, $\delta(G') \geq 2^{m-2}$ by Lemma 3.7. So $|V(G')| \geq 1 + 2^{m-2}$. If this is less than $|V(F')|$, then, since $2^x \geq 2x$ for $x \geq 1$, $|V(F')| \geq 2m$. This is possible only if F' is a matching. But then G' would be obtained by deleting two vertices, a contradiction. □

Proof of Lemma 3.7. May assume G is connected. For a connected $H \subset G$, let $G \odot H$ be the graph obtained from G by contracting all vertices of H into one. Let H be a maximum subgraph of G s.t. $|E(G \odot H)| \geq k \cdot |V(G \odot H)|$.

Each 1-vertex subgraph H is okay. Let $V'(H)$ be the set of neighbors of $V(H)$ in $G - H$. Let $G' = G[V']$. If $d_{G'}(v) \leq k-1$ for some $v \in V'$, then contracting x to H makes at most k edges disappear, contradicting maximality of H. So, $\delta(G') \geq k$. □

Let $h(k) :=$ smallest $\delta(G)$ that implies a subdivision of K_k in G. Clearly, $h(1) = 0$, $h(2) = 1$, $h(3) = 2$. Dirac proved that $h(4) = 3$.

——————————— **Here Lecture 21 ended.**

We know that $h(5) = 6$. In general, $k^2/8 \leq h(k) \leq ck^2$

Hajós conjectured that each graph with chromatic number k is contains a subdivision of K_k.

Theorem 3.8 (Jung, Larman–Many, Th. 7.1.55 in the book)**.** *There is a function $f(k)$ s.t. each $f(k)$-connected graph is k-linked.*

Proof. We know $f(1) = 1$. Will show that $f(k) \leq h(3k)$. By Theorem 3.6, $h(3k) \leq 2^{\binom{3k}{2}}$.

Let G be a $h(3k)$-connected graph. Let H be a subdivision of K_{3k} contained in G with the set Y of branching vertices. Let $X = \{a_1, \dots, a_k, b_1, \dots, b_k\}$. Applying Menger's Theorem, we find $2k$ fully disjoint X, Y-paths with no Y-vertices in the interior.

Among such sets of paths, choose one with the minimum number of edges outside H. Let P_i be the path connecting a_i with some $c_i \in Y$ and let Q_i be the path connecting b_i with some $d_i \in Y$. Let $Y - \{c_1, \dots, c_k, d_1, \dots, d_k\} = \{y_1, \dots, y_k\}$.

Let C_i (resp., D_i) be the path in H connecting y_i with c_i (resp., d_i). Then our paths will be subpaths of walks $a_i P_i C_i D_i Q_i b_i$ for $i \in [k]$. To show that we can choose these paths disjoint we use the choice of our paths (pictures!!). □

Linear bounds on $f(k)$. The record is $f(k) \leq 10k$.

For a graph H, a graph G is *H-linked*, if for any injection $g : V(H) \to V(G)$ for each edge $uv \in E(H)$, G has an $g(u), g(v)$-path P_{uv} s.t. all such paths are internally disjoint.

If M_k denotes a matching with k edges, then k-linked means M_k-linked. The $K_{1,s}$-linked graphs are exactly s-connected graphs.

Theorem 3.9 (Mader, Th. 7.1.59 in the book)**.** *Each graph G with average degree greater than $4k - 4$ has a k-connected subgraph.*

——————————— **Here Lecture 22 ended.**

Proof. For $k \leq 2$, check in class. Let $k \geq 3$.

We prove first another thing: *If*

$$k \geq 3,\ n \geq 2k-1,\ V(G)| = n, \text{ and } |E(G)| > (2k-3)(n-k+1), \tag{13}$$

then G has a k-connected subgraph.

Let G be a smallest counterexample: it satisfies (13), but has no k-connected subgraphs. If $n = 2k - 1$, then

$$|E(G)| > (n-2)(n - \frac{n+1}{2} + 1) = \frac{n(n-1)}{2} - 1.$$

Thus in this case $G = K_{2k-1}$.

Suppose now, $n \geq 2k$. Then by minimality, $\delta(G) \geq 2k-2$. We will show that G is k-connected itself. Indeed, suppose G has a sep. set S with $|S| = k-1$. Let U_1 be the vertex set of a component of $G - S$ and $U_2 = V(G) - S - U_1$. For $i = 1, 2$, let $G_i = G[S \cup U_i]$ and $n_i = |V(G_i)|$.

Since $\delta(G) \geq 2k-2$, $n_i \geq 2k-1$, so by the minimality of G, $|E(G_i)| \leq (2k-3)(n_i - k+1)$, so

$$e(H) \leq (2k-3)(n_1 - k + 1 + n_2 - k + 1 = (2k-3)(n-k+1),$$

contradicting (13). This proves the claim above.

Now we will simply show that each graph G with average degree $a > 4(k-1)$ satisfies (13). Indeed, let $a = 4(k-1) + \epsilon$. Suppose

$$(4k - 4 + \epsilon)\frac{n}{2} \leq (2k-3)(n-k+1).$$

This simply cannot happen. $\square$

Conjecture (Mader, 1972). *For each fixed k for sufficiently large n, every n-vertex graph G with $|E(G)| > (1.5k-2)(n-k+1)$ contains a k-connected subgraph.*

Mader proved the conjecture for $k \leq 6$. He also proved the bound with $1 + 1/\sqrt{2}$ in place of 1.5. Yuster in 2003 proved that if $k \geq 2$ and $n \geq 9k/4$, then each n-vertex graph G with $|E(G)| \geq \frac{193}{120}k(n-k)$ contains a $(k+1)$-connected subgraph. Bernshteyn and A.K. improved $\frac{193}{120}$ to $\frac{19}{12}$ for $n \geq 5k/2$.

——————————— **Here Lecture 23 ended.**

3.3. **Constructive characterizations of 3-connected graphs. Minimally k-connected graphs.** .

Recall characterization of 2-connected graphs using ear decomposition (see the book). It is constructive.

A **vertex k-split** makes H from G by replacing a vertex x with adjacent x_1 and x_2 s.t.
(a) $N_H(x_1) \cup N_H(x_2) = N_G(x) \cup \{x_1, x_2\}$, and
(b) $d_H(x_i) \geq k$ for $i = 1, 2$.
If x_1 and x_2 have no common neighbors, then it is a **disjoint k-split**.

Lemma 3.10. *If G is k-connected and H is a k-split of G, then H is k-connected.*

Proof. Denote $X = \{x_1, x_2\}$. Suppose H has a separating set S with $|S| = k-1$. Then $S \cap X \neq \emptyset$. Also, if $X \subseteq S$, then $(S - X) \cup \{x\}$ is a separating set in G. Thus we may assume $S \cap X = \{x_1\}$.

Let $T = (S - x_1) \cup \{x\}$. Since $|T| = k-1$, $G - T$ is connected. This means $H - S - x_2$ is connected. But out of k neighbors of x_2 at least one is not in S. $\square$

An edge e in a k-connected G is **k-contractible** if G/e is k-connected.

Lemma 3.11 (Contraction Lemma, Tutte, 1961, Lem. 7.2.7 in the book)**.** *Every 3-connected graph $\neq K_4$ has a 3-contractible edge.*

Proof. If xy is not contractible, then there is z s.t. $G' = G - \{x, y, z\}$ is disconnected. Choose x, y, z to maximize the order of the largest component, say H, of $G - \{x, y, z\}$. Let H' be another component of G'. Since G is 3-connected, each of x, y, x has a neighbor in H'. Let u be a neighbor of z in H'.

If uz is contractible, we win. Otherwise, there is a $v \in V(G)$ s.t. $G'' = G - \{v, u, z\}$ is disconnected. If $v \in V(H)$, then it is a cut vertex in $F = G[V(H) \cup \{x, y\}$. Since $v \notin \{x, y\}$ and does not separate x from y, it separates $\{x, y\}$ from $N(z)$ in F. But then $\{v, z\}$ is separating in G!

Thus $v \notin V(H)$. Then a component of $G - \{v, u, z\}$ contains $V(H)$ plus a vertex in $\{x, y\}$, a contradiction. □

The lemma does not hold for k-connected graphs when $k \geq 4$.

Note that each contraction a 3-contractible edge is the inverse of a 3-split. This implies:

Theorem 3.12. *A graph is 3-connected iff it can be obtained from K_4 by a sequence of 3-splits.*

Proof. By Lemma 3.10, each graph obtained from K_4 by a sequence of 3-splits. The other direction is by induction and Lemma 3.11. □

A k-connected graph G is *minimally k-connected* if $G - e$ is not k-connected for any $e \in E(G)$.

Examples.

We will prove the next theorem later, but use soon for another characterization of 3-connected graphs.

Theorem 3.13 (Mader)**.** *Let $k \geq 2$. Every cycle in a minimally k-connected graph contains a vertex of degree k.*

——————————— **Here Lecture 24 ended.**

Lemma 3.14 (Lem. 7.2.13 in the book)**.** *If G is a k-connected graph and $uv \in E(G)$, then*
(a) $G - uv$ is k-connected iff it has no u, v-cut of size $k - 1$;
(b) G/uv is k-connected iff $G - u - v$ is $(k - 1)$-connected.

Proof. For both (a) and (b) one direction is trivial, the other is proved in class. □

Lemma 3.15 (Lem. 7.2.14 in the book)**.** *Let G be a 3-connected graph with $|V(G)| \geq 5$. Suppose $z \in V(G)$ with $d(z) = 3$. Let $t = |E(G[N(z)])|$.*
(a) If $t = 3$ and $u, v \in N(z)$, then $G - uv$ is 3-connected.
(b) If $t \leq 1$, then for some edge $zw \in E(G)$ not in a triangle, G/wz is 3-connected.

Proof. We will think that $N(z) = \{u, v, w\}$.

To prove (a), by Lemma 3.14(a), it is enough to find 3 int.-disjoint u, v-paths. Let w be third neighbor of z and $y \in V(G) - \{z, u, v, w\}$. The 2-connected graph $G - w$ has a $y, \{u, v\}$-fan of size 2. The edges of this fan form a u, v-path P avoiding w and z. So, two other u, v-paths can be u, z, v and u, w, v.

To prove (b), in view of Lemma 3.14(b), we will prove that $G - z - w$ is 2-connected. For this, in turn, we will show that

(14) $$G - z - w \text{ contains a cycle } C \text{ through } u \text{ and } v.$$

Indeed, if (14) holds and $G - z - w$ has a cut vertex x, then there is a component X of $G - w - z - x$ containing neither u nor v. But then X is also a component of $G - w - x$ containing none of u, v and z, a contradiction.

So, we aim at (14). Let $y \in V(G) - \{z, u, v, w\}$. If $uv \in E(G)$, consider a $y, N(z)$-fan of size 3. The paths to u and v in this fan together with edge uv create C. Now we may assume $N(z)$ is independent.

Let y be the neighbor of u on the segment P of C from u to v. Let $V' = V(P) - u - v$. Consider a $y, (V(C) - V')$-fan F of size 3 in G. Since $N(z) \subset V(C) - V'$, $z \notin F$. Let $x \in F \cap (V(C) - V(P))$. We find a cycle through exactly two vertices of $N(z)$ that also goes through x. (**Pictures in class.**)

Lemma 3.16. *Let G be a graph, $z \in V(G)$, $N(z) = \{u, v, w\}$, $vu, vw \in E(G)$, and $uw \notin E(G)$. Then G is 3-connected iff $H := G - z + uv$ is 3-connected.*

Proof. ($\Rightarrow$) Suppose H is not 3-connected. If $|V(H)| = 3$, then $H = K_3$, and so $G = K_4 - e$ not 3-connected. Otherwise, H has a separating $X \subset V(H)$ with $|X| = 2$. Since $\{u, v, w\} - X$ is in one component of $H - X$, X is also separating in G.

($\Leftarrow$) Suppose G is not 3-connected. If $|V(G)| \leq 4$, then $|V(H)| \leq 3$. Suppose $|V(G)| \geq 5$ and let X be a separating set in G with $|X| = 2$. If $X \neq \{z, v\}$, then $\{z, u, v, w\} - X$ is in one component of $G - X$, and so X is also separating in H. Suppose $X = \{z, v\}$ and the size of the component of $G - X$ containing u is not larger than that containing w. Then $\{v, w\}$ is separating in H. □

————————————— **Here Lecture 25 ended.**
————————————- **Lecture 26 was by Bob Krueger.**

Theorem 3.17. *A graph G is 3-connected iff G can be obtained from a wheel by a sequence of adding edges and disjoint 3-splits.*

Proof. ($\Leftarrow$) Immediate by Lemma 3.10.

($\Rightarrow$) We will show that each minimally 3-connected non-wheel G has a contractible edge not in a triangle. Use induction on n. Case $n = 4$ is okay. Let G be a minimum counter-example and $n = |V(G)|$. By Theorem 3.13, G has a vertex z with $d(z) = 3$. If $G[N(z)] = K_3$, by Lemma 3.15(a), G is not minimally 3-connected. If $|E(G[N(z)]| \leq 1$, then by Lemma 3.15(b), G has a contractible edge not in a triangle.

So, suppose $N(z) = \{u, v, w\}$, $vu, vw \in E(G)$, and $uw \notin E(G)$. Let $H = G - z + uv$. By Lemma 3.16, H is 3-connected.

Claim: *H is minimally 3-connected.*

Indeed, suppose $H - e$ is 3-connected. If $e \notin \{uv, vw, uw\}$, then by Lemma 3.16, $G - e$ is also 3-connected, a contradiction.

Suppose now $e = vu$. Since $G - vu$ is not 3-connected, by Lemma 3.14(a), $G - vu$ has a v, u-separating set S with $|S| = 2$. We need $z \in S$. Then $S - z + w$ is v, u-separating in H, as claimed. This also proves that

(*) $\kappa(G - z) = 2$.

The case $e = vw$ is the same. Finally, suppose $e = uw$. Then $H - e = G - z$ and we are done by (*). This proves the claim.

By the claim and IH, either

(A) H is wheel, or
(B) H has an $xy \in E(H)$ s.t. xy is not in a triangle and is 3-contractible.

If (A) holds, then we know G: it is obtained from a wheel by deleting an edge and adding a vertex of degree 3, see pictures in class. In both cases we are done. So suppose (B) holds. Since v, u, w, v is a 3-cycle in H, $xy \notin \{vu, vw, uw\}$. Since H/xy is 3-connected, by Lemma 3.16, G/xy is 3-connected, as claimed. □

Theorem 3.18 (Mader). *Let $k \geq 2$. Every minimally k-connected MULTIgraph contains a vertex of degree k.*

Proof. Let G be a minimally k-connected multigraph. Choose a minimum $X \subset V(G)$ s.t. $|E_G(X, \overline{X})| = k$. Suppose there is $xy \in E(G)$ with $x, y \in X$. Then there is $Z \subset V(G)$ s.t. $Z \cap \{x, y\} = \{x\}$ and

$$|E_G(Z, \overline{Z})| - 1 = |E_{G-xy}(Z, \overline{Z})| = k - 1.$$

Among x, y, choose x so that $X \cup Z \neq V(G)$. Then by submodularity, $|E_G(X \cap Z, \overline{X \cap Z})| = k$. Since $y \notin Z$, $|X \cap Z| < |X|$, a contradiction.

Thus X is independent, so $|X| = 1$. □

Lemma 3.19 (Mader). *Let $k \geq 2$ and let G be a minimally-k-connected graph. Let $a \in V(G)$ with $d(a) \geq k+1$. Let $ax, ay \in E(G)$. Let S be a separating $(k-1)$-set in $G - ax$ and T be a separating $(k-1)$-set in $G - ay$. Then the component of $G - T - ay$ containing y has fewer vertices than the component of $G - S - ax$ containing a.*

——————————— **Here Lecture 27 ended.**

Proof of Theorem 3.13 modulo Lemma 3.19. Suppose that all vertices of a cycle $a_1, \ldots, a_\ell, a_1$ in a minimally-k-connected graph G have degree $\geq k+1$. Let S_i be a separating $(k-1)$-set in $G - a_{i-1}a_i$ ($a_\ell = a_0$). Let A_i be the vertex set of the component of $G - a_{i-1}a_i - S_i$.

By Lemma 3.19 with $a = a_i$, $S = S_i$ and $T = S_{i+1}$, $|A_i| > |A_{i+1}|$ for each i, a contradiction. □

Corollary 3.20 (Bollobás). *Every minimally-k-connected graph with n vertices has $\geq \frac{(k-1)n+2}{2k-1}$ vertices of degree k.*

Proof. Let $S = \{v \in V(G) : d(v) = k\}$. Then

$$2|E(G)| \geq kn + (n - |S|). \qquad (15)$$

By Theorem 3.13, $G - S$ is a forest; so $|E(G - S)| \leq n - |S| - 1$. Thus, using (15),

$$\frac{1}{2}(kn + n - |S|) \leq n - |S| + 1 + k|S|.$$

Solving the inequality for $|S|$, we get the answer. □

Proof of Lemma 3.19. Each of $G - S - ax$ and $G - T - ay$ has exactly two components. Let them be A_X and X (with $a \in A_X$) and A_Y and Y (with $a \in A_Y$). So $V = A_X \cup S \cup X =$

$A_Y \cup T \cup Y$. (PICTURES!!). See which parts are adjacent to which. In particular, since $ax, ay \in E(G)$, $\{x, y\} \cap X \cap Y = \emptyset$!

We want: $|Y| < |A_X|$. The following two imply this: (1) $|Y \cap S| \leq |A_X \cap T|$ and (2) $Y \cap X = \emptyset$.

Claim 1: $|Y \cap S| \leq |A_X \cap T|$.

Proof of Claim 1. If $|Y \cap S| > |A_X \cap T|$, then the set $U = (S - Y) \cup (A_X \cap T)$ satisfies $|U| < |S| = k - 1$. Since $d(a) \geq k + 1$, it has a neighbor not in $U + x + y$. By the picture, $z \in A_X \cap A_Y$. Then $U + a$ separates z from $X \cup Y$, a contradiction.

Claim 2: $Y \cap X = \emptyset$.

Proof of Claim 2. Let $W = (S \cap Y) \cup (T - A_X)$. By Claim 1, $|W| \leq k - 1$. Since $\{x, y\} \cap X \cap Y = \emptyset$, W separates $X \cap Y$ from the rest. □

It is not hard to prove that a multigraph G is 2-edge connected iff it has a strongly connected orientation. (One may use closed-ear decomposition.) Significantly harder is the proof of the following.

Theorem 3.21 (Orientation Theorem, Nash-Williams, 1960, Th. 7.2.29 in the book)**.** *For each $s \geq 1$, a multigraph G has an s-edge-connected orientation iff G is $2s$-edge-connected.*

We need some definitions and a lemma.

A multigraph is *k-edge-connected relative to a vertex z* if each edge-cut apart from maybe $(\{z\}, V - z)$ has at least k edges.

——————————— **Here Lecture 28 ended.**

If $z, u, v \in V(G)$ and $uz, vz \in E(G)$, then the *u, v-shortcut of* z is the graph $G - uz - vz + uv$.

Lemma 3.22 (Shortcut Lemma, Lovász)**.** *Let $k \geq 2$ be even and let z be a vertex of even degree in a multigraph G that is k-edge-connected relative to z. Then for each $u \in N(z)$ there is $v \in N(z)$ s.t. the u, v-shortcut of z is also k-edge-connected relative to z.*

Proof of Theorem 3.21 modulo Lemma 3.22. ($\Rightarrow$) Immediate.

($\Leftarrow$) Use induction on n — the number of vertices. For $n = 2$ — easy. Let G be a counterexample with smallest $n = |V(G)|$ and modulo this, with fewest edges. Then G is minimally $2s$-edge-connected. By Theorem 3.18, G has a vertex z with $d(z) = 2s$. By Lemma 3.22, iteratively find shortcuts of z until in the resulting G' the degree of z is 0. Then $G' - z$ is $2s$-edge-connected. By induction, $G' - z$ has an s-edge-connected orientation. Replace each oriented shortcut edge uv with directed path u, z, v. Lifting these edges does not decrease $d^+(X)$ for any nonempty X not containing z. Also for any nonempty X not containing z, $d^+(X+z)$ after lifting is not less than $d^+(X)$ before lifting. Finally, $d^+(z)$ will be s. □

Proof of Lemma 3.22. Fix $u \in N(z)$. Call $X \subseteq V(G) - z$ *dangerous*, if
(a) $\emptyset \neq X \neq V(G) - z$; (b) $F(X) \leq k + 1$ and (c) $u \in X$.

Claim 1: *If X, Y are dangerous and $X - Y \neq \emptyset \neq Y - X$, then $F(X \cup Y)$ is odd.*

Claim 2: *If X, Y are dangerous, then $F(X \cup Y) \leq k + 1$.*

Claim 3: *If $A \supseteq N(z)$ and $F(A) \leq k + 1$, then $z \in A$.*

Claim 4: *If X, Y are dangerous, then $X \cup Y$ does not contain $N(z)$, and hence is dangerous.*

Let M be the union of all dangerous sets. If $M = \emptyset$, then we can shortcut any uv, even if $u = v$. Let $M \neq \emptyset$. By Claim 4, M is dangerous. By Claim 3, there is $v \in N(z) - M$. Shortcut uv. What remains is to prove the claims. We prove them in the reverse order.

Proof of Claim 4: Suppose $X \cup Y \supseteq N(z)$. By Claim 2, $F(X \cup Y) \leq k+1$. So by Claim 3, $z \in X \cup Y$, contradicting the fact that $z \notin X$ and $z \notin Y$.

Proof of Claim 3: Since $d(z) \geq 2$, if $A \supseteq N(z)$, $F(A) \leq k+1$ and $z \in A$, then $F(A+z) = F(A) - d(z) \leq (k+1) - 2 < k$, a contradiction.

Proof of Claim 2: If $X \subseteq Y$ or $Y \subseteq X$, this is trivial. Suppose $X - Y \neq \emptyset \neq Y - X$. By submodularity of F,

$$F(X \cap Y) + F(X \cup Y) \leq F(X) + F(Y) \leq 2(k+1).$$

Hence $F(X \cup Y) \leq 2(k+1) - F(X \cap Y) \leq 2k+2$. So by Claim 1, $F(X \cup Y) \leq k+1$.

———————————— **Here Lecture 29 ended.**

Proof of Claim 1: Since $uz \in E(X \cap Y, \overline{X \cup Y})$,

$$2(k+1) \geq F(X) + F(Y) = F(X-Y) + F(Y-X) + 2|E(X \cap Y, \overline{X \cup Y})| \geq k+k+2.$$

So, we have all equalities here; in particular, $F(X) = F(Y) = k+1$ and $F(X-Y) = F(Y-X) = k$. Since $F(Y) + F(X-Y) \equiv F(X \cup Y) (\mod 2)$, the claim follows. □

Theorem 3.23 (Győri, Lovász, Th. 7.2.23 in the book)**.** *An n-vertex graph G is k-connected iff $n \geq k+1$ and for all distinct $v_1, \ldots, v_k \in V(G)$ and any positive integers $n_1, \ldots, n_k$ s.t. $n_1 + \ldots + n_k = n$, there is a partition $V(G) = V_1 \cup \ldots \cup V_k$ s.t. for each $1 \leq i \leq k$,*

(a) $G[V_i]$ is connected, (b) $v_i \in V_i$, and (c) $|V_i| = n_i$.