2. Lecture notes: Reconstruction

A model of reconstructing a whole object from its parts is Graph reconstruction. For a graph $G$, a card or vds is a subgraph $G - v$ for some $v \in V(G)$. The deck is the set of all cards of a graph. A graph is reconstructible if no other graph has the same deck.

**Examples:** A graph with 2 vertices and a graph with 5 vertices.

**Reconstruction Conjecture:** Every graph with at least 3 vertices is reconstructible. Proved for narrow classes.

A graph parameter is reconstructible if it can be computed from the deck when $n > 2$.

A class $\mathcal{G}$ of graphs is recognizable if the property of membership in $\mathcal{G}$ is reconstructible.

**Examples.**

Let $s_Q(G) = \#$ of copies of $Q$ in $G$,

$s_Q^*(G) = \#$ of induced copies of $Q$ in $G$,

$s_Q(G, v) = \#$ of copies of $Q$ in $G$ containing $v$,

$s_Q^*(G, v) = \#$ of induced copies of $Q$ in $G$ containing $v$.

**Theorem 2.1** (Kelly’s Lemma, Kelly, 1957, Lem. 6.3.6 in the book). If $n > 2$, $v \in V(G)$ and $|V(Q)| < |V(G)|$, then all of $s_Q(G)$, $s_Q^*(G)$, $s_Q(G, v)$ and $s_Q^*(G, v)$ are reconstructible. In particular, the degrees sequence and the number of edges are reconstructible.

**Proof.** $s_Q(G) = \sum_{v \in V(G)} \frac{s_Q(G - v)}{n - |Q|}$, $s_Q(G, v) = s_Q(G) - s_Q(G - v)$. □

**Corollary:** Regular graphs are reconstructible.

**Theorem 2.2** (Kelly, 1957, Th. 6.3.13 in the book). Disconnected graphs with at least 3 vertices are reconstructible.

**Proof.** First we show that the class of disconnected graphs is recognizable. For this, observe that a graph $G$ is connected iff at least two of its vds are connected.

Now, if some card of a disconnected graph is connected, then this vertex is isolated and we see the rest of the graph in the card. If none of the cards is connected, choose a largest component over all cards, say $M$. Fix any subgraph $L$ of $M$ with $|V(L)| = |M| - 1$. Among the cards with the fewest copies of $M$, choose one with the most copies of $L$-components. Then we know all. □

A more complicated theorem is about reconstruction of trees. We need some notions and claims.

Recall that each tree has one or two adjacent centers. The branches of a bicentral tree are the component obtained by deleting the central edges. The branches of an unicentral tree $T$ with center $c$ are the components of $T - c$ with the added $c$ adjacent to its neighbor in $T$ in this component. They are rooted trees with the root in the center.

**Examples.**
When \( \Delta(G) > 2 \), \( \alpha(v) \) denotes the distance from \( v \) to the closest vertex of degree at least 3. A \textit{peripheral} vertex is a vertex with largest eccentricity. An \textit{arm} in a tree is a branch containing a peripheral vertex.

\textbf{Lemma 2.3.} Let \( n \geq 3 \).

(a) Trees, paths and trees of diameter \( d \) are recognizable.

(b) For a tree \( T \), the set \( \{ \alpha(v) \}_{v \in V(T)} \) is reconstructible.

\textbf{Proof.} Each tree is a connected graph with \( n - 1 \) edges. A path is a tree with max degree 2. If a tree is not a path, then we see the longest path in a card. This proves (a).

For (b), if \( T \) is a path, then \( \alpha(v) \) is not defined for all \( v \). Suppose not. For every vertex of degree at least 3, we know this, and this means \( \alpha(v) = 0 \). Suppose \( d(v) = 2 \).

Let \( Y_k \) be tree with \( k + 3 \) vertices obtained from the path with \( k + 2 \) vertices by duplicating one leaf. For each \( k < n - 3 \) and each \( v \) we know \( s_{Y_k}(T,v) \). The least \( k \) such that \( s_{Y_k}(T,v) > 0 \) (if exists) is \( \alpha(v) \). If such \( k \) does not exist, then since \( T \) is not a path, \( \alpha(v) = n - 3 \). \( \square \)

\textbf{Theorem 2.4} (Kelly, 1957, Th. 6.3.19 in the book.). \textit{Trees with at least 3 vertices are reconstructible.}

\textbf{Proof.} Let a deck \( D \) be given. By Lemma 2.3(a), we may assume that \( G \) is a tree distinct from the path. And we know its diameter. Since peripheral vertices are those that belong to a path of length \( \text{diam}(G) \) and have degree 1, we know the cards of peripheral vertices. Let \( P \) be this set of cards.

Call a tree \textit{special} if it has exactly two branches, and one is a path. If \( G - v \in P \), then the arm containing \( v \) is a path iff \( \alpha(v) \geq \frac{\text{diam}(G)}{2} \). If in addition \( G \) is special, then \( \alpha(v) > \frac{\text{diam}(G)}{2} \).

Thus

\[ G \text{ is special } \iff P \text{ has } G - v \text{ with } \alpha(v) > \frac{\text{diam}(G)}{2}. \]

So we can recognize whether \( G \) is special. If yes, then reconstruct \( G \) from \( G - v \in P \) by appending \( v \) to any path arm of \( G - v \). So, suppose not.

Let \( Q = \{ G - v : \text{diam}(G - v) = \text{diam}(G) \text{ and } d(v) = 1 \} \). We now show that

\[ \forall \text{ arm } A \text{ there is a leaf } w \notin A \text{ s.t. } G - w \in Q. \]

Indeed, if for each leaf \( w \notin A \), \( \text{diam}(G - w) < \text{diam}(G) \), then only one leaf is not in \( A \); thus \( G \) is special.

Let \( A \) be a largest arm. By (2) some \( G - w \in Q \) contains \( A \). Preserving diameter preserves the center. So, \( A \) is an arm in \( G - w \). Thus from \( Q \) we see all largest arms of \( G \).

\textbf{Case 1:} \( A \) is a path arm. Then each arm in cards in \( Q \) is a path arm. Take a connected card with the fewest path arms and append \( v \) to a slightly shorter branch that is a path.

\textbf{Case 2:} \( A \) is not a path. Then there is a leaf \( u \in A \) s.t. \( G - u \in Q \). Let \( L = A - u \). Then \( L \) is an arm in \( G - u \), so in a card \( C \in Q \) with the fewest arms isomorphic \( A \) and most cards isomorphic \( L \) we replace one \( L \) with \( A \). \( \square \)

\begin{center}
Here Lecture 13 ended.
\end{center}

\textbf{Theorem 2.5} (Tutte, 1976, Th. 6.3.21 in the book.). \textit{For } \( n \geq 3 \) \textit{ a graph } \( G \) \textit{ with } \( n \) \textit{ vertices, the parameters below are reconstructible.}

(A) \( s_Q \) if \( Q \) is a spanning disconnected subgraph with \( \delta(Q) \geq 1 \).
(B) For \( k \geq 2 \), the number of spanning connected subgraphs of \( G \) whose blocks are \( B_1, \ldots, B_k \).

(C) The number of 2-connected spanning subgraphs of \( G \) with \( m \) edges.

Note: we do not see these subgraphs in the cards.

**Proof of (A).** Suppose \( Q_1, \ldots, Q_k \) are the components of \( Q \).

For a graph \( H \), define \( b_Q(H) = \# \) of ways to express \( H \) as the union of \( Q_1, \ldots, Q_k \).

**Example:** \( Q_1 = K_3, Q_2 = P_3, Q_3 = K_2, H_1 = K_4 - e, H_2 = K_4 \). Then \( b_Q(H_1) = 2(5 + 4) = 18 \) and \( b_Q(H_2) = 12 \).

Important equality is:

\[
\prod_{i=1}^{k} s_{Q_i}(G) = \sum_{H \subseteq G, \delta(H) \geq 1} b_Q(H)s_H(G).
\]

Given any \( H \), we know \( b_Q(H) \). If \( |V(H)| \leq n - 1 \), then we know \( s_H(G) \). So, from (3) we know \( s_Q(G) \).

**Proof of (B).** Suppose \( B = \{ B_1, \ldots, B_k \} \) is the list of blocks, and \( n_i = |V(B_i)| \). Each connected graph with blocks \( B_1, \ldots, B_k \) has \( \sum_{i=1}^{k} n_i - k + 1 \) vertices.

For a graph \( H \), define \( b_B(H) = \# \) of ways to express \( H \) as the union of \( B_1, \ldots, B_k \). Again (3) with \( B \) in place of \( Q \) holds. We know: (a) \( b_B(H) \) for all \( H \), (b) \( s_H(G) \) when \( |V(H)| < n \) or \( H \) is disconnected.

Let \( S \) be the class of connected spanning subgraphs of \( G \) whose blocks are \( B_1, \ldots, B_k \). So, unknown are the values of \( s_H(G) \) when \( H \in S \). We do not find each of them, but want to find \( \sum_{H \in S} s_H(G) \). We know that for all such \( H \), \( b_B(H) \) is the same: it is 1 when all \( B_i \) are distinct, and otherwise it is \( (m_1!) \ldots (m_j!) \) when they form \( j \) isomorphism classes.

**Proof of (C).** There are \( \binom{|E(G)|}{m} \) subgraphs of \( G \) with \( m \) edges. By Kelley’s Lemma we know the number of them with isolated vertices. By (A), we know the number of other disconnected subgraphs with \( m \) edges. By (B), we know the number of connected subgraphs with \( m \) edges and with cut vertices. □

**Corollary.** The number of hamiltonian cycles and the number of spanning trees in a graph are reconstructible.

Bollobás result on 3 cards.

Edge-reconstruction, examples with 3 edges.

**Edge-Reconstruction Conjecture** (Harary, 1964): Every graph with more than 3 edges is edge-reconstructible.

Here Lecture 14 ended.

**Lemma 2.6** (Edge-Kelly Lemma). Let \( m \geq 4 \). If \( |E(G)| = m > |E(Q)| \), then \( s_Q(G) \) is reconstructible.

**Proof.** The same as for Kelly Lemma. □

Let \( s_Q'(G) \) be \# of injections \( f : V(Q) \to V(G) \) s.t. edges of \( Q \) go to edges of \( G \).

Then \( s_Q'(G) = a(Q) \cdot s_Q(G) \), where \( a(Q) \) is the number of automorphisms of \( Q \).
Theorem 2.7 (Lovász, 1972), Th. 6.3.31 in the book. Let $G$ be an $n$-vertex graph with $m$ edges. If $m > \frac{1}{2} \binom{n}{2}$, then $G$ is edge-reconstructible.

**Proof.** We look at $n$-vertex graphs. For a graph $Q$, let $\mathcal{Q}(Q)$ be the set of all $2^{|E(Q)|}$ spanning subgraphs of $Q$. By inclusion-exclusion, for each $G$,

$$s'_Q(G) = \sum_{F \in \mathcal{Q}(Q)} (-1)^{|E(F)|} s'_F(G).$$

Suppose $n$-vertex $m$-edge graph $G_1$ has the same edge deck as $G$. By (4),

$$s'_{G_1}(\overline{G}) = \sum_{F \in \mathcal{Q}(G_1)} (-1)^{|E(F)|} s'_F(G)$$

and

$$s'_{G}(\overline{G}) = \sum_{F \in \mathcal{Q}(G)} (-1)^{|E(F)|} s'_F(G).$$

The terms in RHSs of (5) and (6) containing $F$ distinct from $G_1$ and $G$ are the same. Also, both LHSs are zeros, since $|E(G)| > |E(\overline{G})|$. So $s'_{G_1}(\overline{G}) = s'_G(\overline{G}) > 0$, which means $G_1 = G_2$. □

For a spanning subgraph $R$ of $Q$, let $s'_{RQ}(G)$ denote the number of injections $f : V(Q) \to V(G)$ s.t. the edges in $R$ map into edges of $G$ and the edges in $Q - E(R)$ map into non-edges of $G$.

Theorem 2.8 (Nash-Williams, 1976), Th. 6.3.33 in the book). If a graph $G$ with at least 4 edges has a spanning subgraph $R$ satisfying one of the properties below, then $G$ is edge-reconstructible.

1) $s'_{RG}(H) = s'_{RG}(G)$ for all $H$ with the same edge deck as $G$.
2) $|E(G)| - |E(R)|$ is even and $s'_{RG}(G) = 0$.

Corollary 2.9 (Müller, 1977), Cor. 6.3.34 in the book). Every graph $G$ with $n \geq 4$ vertices and at least $1 + \log_2(n!)$ edges is edge-reconstructible.

**Proof of Corollary 2.9 modulo Th. 2.8.** Let $m = |E(G)| \geq 1 + \log_2(n!)$. $G$ has $2^{n-1}$ spanning subgraphs $R$ s.t. $m - |E(R)|$ is even.

There are $n!$ injections $V(G) \to V(G)$; they preserve at most $n!$ sets $R$. If $2^{n-1} > n!$, then some $R$ is never preserves, that is, $s'_{RG}(G) = 0$. Apply part 2) of Th. 2.8. □

Here Lecture 15 ended.

**Proof of Th. 2.8.** For a spanning subgraph $R$ of $G$, let $\mathcal{Q}(R)$ be the set of spanning subgraphs of $G$ containing $R$.

For every graph $F$, $s'_R(F) = \sum_{P \in \mathcal{Q}} s'_{P,G}(F)$. So, by Inclusion-Exclusion,

$$s'_{RG}(F) = \sum_{P \in \mathcal{Q}} (-1)^{|E(P)| - |E(R)|} \ s'_{P}(F).$$

Let $G'$ have the same edge deck as $G$. Consider $s'_{RG}(G)$ and $s'_{RG}(G')$. By edge-Kelly Lemma, almost all terms in RHS of (7) coincide, so

$$s'_{RG}(G) - s'_{RG}(G') = (-1)^{|E(G)| - |E(R)|} (s'_G(G) - s'_{G'}(G')).$$
Let $\kappa'(r, G)$ be the minimum # of edges whose deletion makes some $v \in V(G)$ unreachable from $r$.

For $X \subset V(G)$, let $F(X) = \#$ of edges entering $X$. So

$$\kappa'(r, G) = \min\{F(X) : X \neq \emptyset, r \notin X\}.$$ 

Let $b(r, G) = \max \#$ of edge-disjoint $r$-branchings in $G$.

**Theorem 3.1** (Edmonds, 1973, Th. 7.1.37 in the book). For each digraph $G$ and each $r \in V(G)$, $b(r, G) = \kappa'(r, G)$.

**Proof.** Let $k = \kappa'(r, G)$. The fact $b(r, G) \leq k$ is evident. We prove $b(r, G) \geq k$ by induction on $k$. The case $k = 1$ is clear..

3. **Connectivity**

3.1. **New min-max theorems.** Definitions and examples. Recollecting Menger Theorems, Expansion Lemma.

An $r$-branching in a digraph is an out-tree rooted at $r$.

Let $\kappa'(r, G)$ be the minimum # of edges whose deletion makes some $v \in V(G)$ unreachable from $r$.

For $X \subset V(G)$, let $F(X) = \#$ of edges entering $X$. So

$$\kappa'(r, G) = \min\{F(X) : X \neq \emptyset, r \notin X\}.$$ 

Let $b(r, G) = \max \#$ of edge-disjoint $r$-branchings in $G$.

**Theorem 3.1** (Edmonds, 1973, Th. 7.1.37 in the book). For each digraph $G$ and each $r \in V(G)$, $b(r, G) = \kappa'(r, G)$.

**Proof.** Let $k = \kappa'(r, G)$. The fact $b(r, G) \leq k$ is evident. We prove $b(r, G) \geq k$ by induction on $k$. The case $k = 1$ is clear.

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For the induction step, we will find an $r$-branching $T$ s.t. $\kappa'(r, G - E(T)) \geq k - 1$.

**Claim 1:** For all $U, W \subseteq V(G)$, $F(U) + F(W) \geq F(U \cup W) + F(U \cap W)$. Proof in class.

A partial $r$-branching (p.b. for short) is an out-tree with root $r$. A p.b. is good if $\kappa'(r, G - E(B)) \geq k - 1$.

A p. b. with one edge is good. Let $B$ be a largest good p.b. If for every $W \subseteq V(G)$ s.t. (a) $r \notin B$ and (b) $W \not\subseteq B$ we have $F_{G-E(B)}(W) \geq k$, then adding any edge from $V(B)$ to $V(G) - V(B)$ we get a good p.b. contradicting the choice of $B$.

So, choose a minimum $U \subset V(G)$ satisfying (a) and (b) s.t. $F_{G-E(B)}(U) = k - 1$. Since no edges entering $U - V(B)$ were deleted, $F_{G-E(B)}(U - V(B)) \geq k$.

Draw a picture!!

But $F_{G-E(B)}(U) = k - 1$. So there is $xy \in E(G)$ s.t. $x \in V(B) \cap U$ and $y \in U - V(B)$. Let $B' = B + xy$. By the maximality of $B$, $\kappa'(r, G - E(B')) \leq k - 2$. This means there is $W \subseteq V - r$ s.t.

$$F_{G-E(B')}(W) \leq k - 2.$$ 

This in turn means $F_{G-E(B)}(W) = k - 1$ and $xy$ enters $W$, i.e. $x \notin W$ and $y \in W$. In particular, $U \cap W \neq U$. By Claim 1,

$$F_{G-E(B)}(W \cap U) + F_{G-E(B)}(W \cup U) \leq F_{G-E(B)}(W) + F_{G-E(B)}(U) = 2(k - 1).$$

It follows that $F_{G-E(B)}(W \cap U) = F_{G-E(B)}(W \cup U) = k - 1$. This contradicts the choice of $U$. ∎

**Corollary 3.2** (Cor. 7.1.38 in the book). For each digraph $G$ and any $r \in V(G)$, TFAE:

(A) $G$ has $k$ pairwise edge-disjoint $r$-branchings.

(B) $\kappa'(r, G) \geq k$. 


(C) For each \( s \in V(G) - r \), \( \exists k \) pairwise edge-disjoint \( r, s \)-paths.

(D) The underlying undirected \( H \) has \( k \) pairwise edge-disjoint spanning trees whose union \( G' \) is s.t. each vertex apart from \( r \) is entered by exactly \( k \) edges.

Proof. (A) \( \Rightarrow \) (C) Evident. (C) \( \Rightarrow \) (B) All \( r, s \)-paths should be broken.

(B) \( \Rightarrow \) (A) Theorem 3.1. (A) \( \Rightarrow \) (D) Evident.

(D) \( \Rightarrow \) (B) Let \( U \subseteq V - r \). Each spanning tree has at most \( |U| - 1 \) edges inside \( U \), so \( |E_{G'}(U)| \leq k(|U| - 1) \). But altogether there are \( k|U| \) edges entering the vertices in \( U \). \( \square \)

Theorem 3.3 (Seymour, 1977, Th. 7.1.39 in the book). Theorem 3.1 implies the edge local directed version of Menger’s Theorem.

Proof. Let \( x, y \in V(G) \) and \( k = \kappa'(x, y) \). By the definition of \( k \), for each \( U \subseteq V(G) - x \) with \( y \in U \), \( F_G(U) \geq k \).

Let \( G' \) be obtained from \( G \) by adding \( k \) edges \( yz \) for each \( z \in V(G) - x - y \). Then \( F_{G'}(U) \geq k \) for each \( U \) not containing \( x \). So by Theorem 3.1, \( G' \) has \( k \) edge-disjoint \( x \)-branchings. Each of them contains an \( x, y \)-path, and this path is contained in \( G \). \( \square \)

— Here Lecture 17 ended.

A dicut in a digraph \( G \) is an ordered partition \([S, \overline{S}]\) of \( V(G) \), s.t. \( G \) has no edges from \( S \) to \( \overline{S} \).

By definition, a digraph is strongly connected iff it has no dicuts.

If the underlying undirected graph \( G \) is connected, then each dicut \([S, \overline{S}]\) has edge(s) from \( S \) to \( \overline{S} \). Such edges we will call the edges of \([S, \overline{S}]\).

If we add to \( G \) a set \( L \) of directed edges s.t. for each dicut \([S, \overline{S}]\), \( L \) contains an edge from \( S \) to \( \overline{S} \), then \( G + L \) is strongly connected. Certainly, for a digraph \( G \) with the underlying undirected graph \( G \) connected, the number of edges in such \( L \) must be at least the maximum number \( m(G) \) of pairwise disjoint dicuts in \( G \). We will prove a theorem by Lucchesi and Younger that one can find such \( L \) of size \( m(G) \) with the property that each edge in \( L \) is a reversed edge from \( G \). It was a conjecture by Younger and Robertson.

For the proof, we need a technical lemma by Lovász.

Lemma 3.4 (Lovász, 1976), Lem. 7.1.46 in the book). Let \( G \) be a digraph with at most \( k \) pairwise disjoint dicuts. If \( D_1, \ldots, D_\ell \) are dicuts that together cover each edge of \( G \) at most twice, then \( \ell \leq 2k \).

Note that the same dicut may appear twice among \( D_1, \ldots, D_\ell \).

Theorem 3.5 (Lucchesi and Younger, 1978, Th. 7.1.47 in the book). For a digraph \( G \) with the underlying undirected graph \( G \) connected, the minimum number of edges in a set covering all dicuts equals the maximum number \( m(G) \) of pairwise disjoint dicuts in \( G \).

Proof modulo Lemma 3.4. By induction on \( m(G) \). If \( m(G) = 0 \), the claim is trivial. Suppose the theorem holds for all \( G' \) with \( m(G') \leq k - 1 \). Let \( G \) be any digraph with \( m(G) = k \).

Definitions of subdivisions and contractions: \( G \oplus e \) and \( G/e \).

Let \( D = ([S, \overline{S}] \) be a dicut in a set of \( k \) pairwise disjoint dicuts in \( G \). We subdivide edges of \( D \) one by one until subdividing any other edges from \( D \) would increase the number of
pairwise disjoint dicuts. Suppose the resulting digraph is $H$, and $e$ is an edge in $D$ s.t. $H \oplus e$ has $k + 1$ disjoint dicuts, say $D_1, \ldots, D_{k+1}$. We can consider those as dicuts in $H$ such that $D_1$ and $D_2$ share $e$, but in each other pair of dicuts the dicuts are disjoint.

Consider $H' = H/e$. If $H'$ has only $k - 1$ disjoint dicuts, then $G/e$ also has at most $k - 1$ disjoint dicuts, hence by induction has a set $S$ of $k - 1$ edges covering all dicuts in $G/e$. But then $S$ covers all dicuts in $G$ that do not contain $e$. Thus $S + e$ covers all dicuts in $G$, a contradiction.

Hence $H'$ has $k$ disjoint dicuts, say $C_1, \ldots, C_k$. Those are disjoint dicuts in $H$ not containing $e$. Then $\{C_1, \ldots, C_k, D_1, \ldots, D_{k+1}\}$ is a set of $2k + 1$ dicuts in $H$ contradicting Lemma 3.4. □

For the proof of Lemma 3.4, we will need some notation.

Sets $A$ and $B$ in a universe $U$ are crossing if all of $A \cap B, A - B, B - A$ and $\overline{A} \cup B$ are non-empty. A family of sets is laminar if no two members are crossing.

**Proof of Lemma 3.4.** Let $k \geq 1$. Suppose a digraph $G$ with at most $k$ pairwise disjoint dicuts has dicuts $D_1, \ldots, D_{2k+1}$ that together cover each edge of $G$ at most twice. Let $D_i = (S_i, \overline{S}_i)$ for $1 \leq i \leq 2k + 1$.

Choose such a set with the maximum $\sum_{i=1}^{2k+1} |S_i|^2$. We claim that

\[ \text{the family } S = \{S_1, \ldots, S_{2k+1}\} \text{ is laminar.} \tag{9} \]

Indeed, if $S_1$ and $S_2$ cross, replace $D_1$ and $D_2$ with pairs $D'_1 = (S_1 \cap S_2, \overline{S_1} \cap \overline{S_2})$ and $D'_2 = (S_1 \cup S_2, \overline{S_1} \cup \overline{S_2})$. We check (using pictures!) that (a) $D'_1$ are dicuts, (b) each edge of is covered at most twice by $D'_1, D'_2, D_3, \ldots, D_{2k+1}$, and
(c) $|S_1 \cap S_2|^2 + |S_1 \cup S_2|^2 \geq |S_1|^2 + |S_2|^2$. This proves (9).

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**Here Lecture 18 ended.**

Consider the auxiliary graph $H$ with $V(H) = U = \{D_1, \ldots, D_{2k+1}\}$, and $D_i D_j \in E(H)$ iff $D_i$ and $D_j$ share an edge. By the definition of $k$, $\alpha(H) \leq k$. We will prove that

\[ H \text{ is bipartite.} \tag{10} \]

That would imply $|V(H)| \leq 2\alpha(H) \leq 2k$, a contradiction.

So suppose $C = D_1, \ldots, D_m$. $D_1$ is an odd cycle in $H$. If some $D_i$ appears twice in $C$, then the edges of $D_i$ do not belong to other $D_j$s, a contradiction. So, all $D_1, \ldots, D_m$ are distinct, hence all $S_1, \ldots, S_m$ are distinct.

Since $D_i \cap D_{i+1} \neq \emptyset$, $S_i \cap \overline{S}_{i+1} \neq \emptyset$ and $\overline{S}_i \cap \overline{S}_{i+1} \neq \emptyset$. Since $\{S_i, S_{i+1}\}$ is non-crossing,

\[ \text{either } S_i \subset S_{i+1} \text{ or } S_i \supset S_{i+1}. \tag{11} \]

Since $m$ is odd, the condition cannot alternate all the time. So, we may assume

\[ S_m \subset S_1 \subset S_2. \tag{12} \]

Let $j$ be the largest index s.t. $S_1$ contains neither $S_j$ nor $\overline{S}_j$. By (12), $j$ is well defined and $j \leq m - 1$.

By (9), (*)& either $S_1 \subset S_j$ or $S_1 \subset \overline{S}_j$. Let $e = xy \in D_j \cap D_{j+1}$.

Pictures!!

Rewriting (*), we have
(a) Either $S_1 \subset S_j$ or $S_1 \cap S_j = \emptyset$. Similarly,
Lemma 3.7

Also, by (a) we have two subcases.

Case 1: $S_j \subset S_{j+1}$. By (a) we have two subcases.

Case 1.1: $S_j \not\supset S_{j+1}$ by (b), $S_j \supset \overline{S}_{j+1}$. But $S_1$ does not contain $y$.

Case 1.2: $S_j \not\supset S_{j+1}$. (Picture!) Then $S_1 \not\supset S_{j+1}$, so by (b), $S_1 \supset \overline{S}_{j+1}$. But $S_1$ does not contain $x$.

Case 2: $S_j \supset S_{j+1}$. By (a) we have two subcases.

Case 2.1: $S_j \not\supset S_{j+1}$ by (b), $S_1 \supset \overline{S}_{j+1}$. But then $e \in D_j \cap D_{j+1} \cap D_1$.

Case 2.2: $S_j \not\supset S_{j+1}$ by (b) and (c) contradicting (b). □

3.2. On $k$-linked graphs. A graph $G$ with at least $2k$ vertices is $k$-linked, if for any distinct $a_1, \ldots, a_k, b_1, \ldots, b_k \in V(G)$, there are $k$ disjoint paths $P_1, \ldots, P_k$ s.t. $V_i, P_i$ is an $a_i, b_i$-path.

An example of a 5-connected but not 2-linked graph.

Jung: each non-planar 4-connected graph is 2-linked. So, each 6-connected graph is 2-linked.

For each fixed $k$, there is an $O(n^3)$-algorithm checking whether an $n$-vertex $G$ is $k$-linked. For general $k = \text{NP-hard}$.

Before continuing of $k$-linked graphs, we digress on subdivisions. Recall the definition! Also, $F$-subdivisions.

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Here Lecture 20 ended.

Theorem 3.6 (Mader, Thomassen, Th. 7.1.53 in the book). Let $F$ have $m$ edges and no isolated vertices. If a graph $G$ has at least $|V(F)|$ vertices and $\delta(G) \geq 2^{m-1}$, then $G$ contains an $F$-subdivision.

We will use the lemma below:

Lemma 3.7 (Mader, Thomassen, Lem. 7.1.52 in the book). If $\delta(G) \geq 2k$, then $G$ contains vertex disjoint subgraphs $G'$ and $H$ s.t. (1) $\delta(G') \geq k$, (2) each $v \in V(G')$ has a neighbor in $H$ and (3) $H$ is connected.

Proof of Theorem 3.6 modulo Lemma 3.7. By induction on $m$. Check for $m = 1, 2$. Suppose $m \geq 3$ and the theorem is proved for $m - 1$.

If there is $xy \in E(F)$ with $d(x) = d(y) = 1$, then $F' = F - x - y$. In this case, choose any edge $uv \in E(G)$ and let $G' = G - u - v$. Otherwise, let $G'$ and $H$ satisfy Lemma 3.7, and define $F'$ as follows. If there is $xy \in E(F)$ with $d(x) \geq 2$ and $d(y) = 1$, then let $F' = F - y$. Otherwise $\delta(F) \geq 2$. Take any $xy \in E(F)$ and let $F' = F - xy$.

We claim that $G'$ satisfies conditions for $F'$. Indeed, if $d(x) = d(y) = 1$, then $\delta(G') \geq \delta(G) - 2 \geq 2^{m-1} - 2 \geq 2^{m-2}$. Also in this case $|V(G')| = |V(G)| - 2 \geq |V(F)| - 2$.

In other cases, $\delta(G') \geq 2^{m-2}$ by Lemma 3.7. So $|V(G')| \geq 1 + 2^{m-2}$. If this is less than $|V(F')|$, then, since $2^x \geq 2x$ for $x \geq 1$, $|V(F')| \geq 2m$. This is possible only if $F'$ is a matching. But then $G'$ would be obtained by deleting two vertices, a contradiction. □
Proof of Lemma 3.7. May assume \( G \) is connected. For a connected \( H \subset G \), let \( G \odot H \) be the graph obtained from \( G \) by contracting all vertices of \( H \) into one. Let \( H \) be a maximum subgraph of \( G \) s.t. \(|E(G \odot H)| \geq k \cdot |V(G \odot H)|\).

Each 1-vertex subgraph \( H \) is okay. Let \( V'(H) \) be the set of neighbors of \( V(H) \) in \( G - H \). Let \( G' = G[V'] \). If \( d_{G'}(v) \leq k - 1 \) for some \( v \in V' \), then contracting \( x \) to \( H \) makes at most \( k \) edges disappear, contradicting maximality of \( H \). So, \( \delta(G') \geq k \). \( \square \)

Let \( h(k) := \) smallest \( \delta(G) \) that implies a subdivision of \( K_k \) in \( G \). Clearly, \( h(1) = 0 \), \( h(2) = 1 \), \( h(3) = 2 \). Dirac proved that \( h(4) = 3 \).

——— Here Lecture 21 ended. ————

We know that \( h(5) = 6 \). In general, \( k^2/8 \leq h(k) \leq ck^2 \)

Hajós conjectured that each graph with chromatic number \( k \) is contains a subdivision of \( K_k \).

Theorem 3.8 (Jung, Larman–Many, Th. 7.1.55 in the book). There is a function \( f(k) \) s.t. each \( f(k) \)-connected graph is \( k \)-linked.

Proof. We know \( f(1) = 1 \). Will show that \( f(k) \leq h(3k) \). By Theorem 3.6, \( h(3k) \leq 2^{3k} \).

Let \( G \) be a \( h(3k) \)-connected graph. Let \( H \) be a subdivision of \( K_{3k} \) contained in \( G \) with the set \( Y \) of branching vertices. Let \( X = \{a_1, \ldots, a_k, b_1, \ldots, b_k\} \). Applying Menger’s Theorem, we find \( 2k \) fully disjoint \( X, Y \)-paths with no \( Y \)-vertices in the interior.

Among such sets of paths, choose one with the minimum number of edges outside \( H \). Let \( P_i \) be the path connecting \( a_i \) with some \( c_i \in Y \) and let \( Q_i \) be the path connecting \( b_i \) with some \( d_i \in Y \). Let \( Y - \{c_1, \ldots, c_k, d_1, \ldots, d_k\} = \{y_1, \ldots, y_k\} \).

Let \( C_i \) (resp., \( D_i \)) be the path in \( H \) connecting \( y_i \) with \( c_i \) (resp., \( d_i \)). Then our paths will be subpaths of walks \( a_i P_i C_i D_i Q_i b_i \) for \( i \in [k] \). To show that we can choose these paths disjoint we use the choice of our paths (pictures!!). \( \square \)

Linear bounds on \( f(k) \). The record is \( f(k) \leq 10k \).

For a graph \( H \), a graph \( G \) is \( H \)-linked, if for any injection \( g : V(H) \to V(G) \) for each edge \( uv \in E(H) \), \( G \) has an \( g(u), g(v) \)-path \( P_{uv} \) s.t. all such paths are internally disjoint.

If \( M_k \) denotes a matching with \( k \) edges, then \( k \)-linked means \( M_k \)-linked. The \( K_{1,s} \)-linked graphs are exactly \( s \)-connected graphs.

Theorem 3.9 (Mader, Th. 7.1.59 in the book). Each graph \( G \) with average degree greater than \( 4k - 4 \) has a \( k \)-connected subgraph.

——— Here Lecture 22 ended.